

# DRINFELD ORBIFOLD ALGEBRAS

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**ABSTRACT.** We define Drinfeld orbifold algebras as filtered algebras deforming the skew group algebra (semi-direct product) arising from the action of a finite group on a polynomial ring. They simultaneously generalize Weyl algebras, graded (or Drinfeld) Hecke algebras, rational Cherednik algebras, symplectic reflection algebras, and universal enveloping algebras of Lie algebras with group actions. We give necessary and sufficient conditions on defining parameters to obtain Drinfeld orbifold algebras in two general formats, both algebraic and homological. We explain the connection between Hochschild cohomology and a Poincaré-Birkhoff-Witt property explicitly (using Gerstenhaber brackets). We also classify those deformations of skew group algebras which arise as Drinfeld orbifold algebras and give applications for abelian groups.

## 1. INTRODUCTION

Results in commutative algebra are often obtained by an excursion through a larger, noncommutative universe. Indeed, interesting noncommutative algebras often arise from deforming the relations of a classical commutative algebra. Noncommutative algebras modeled on groups acting on commutative polynomial rings serve as useful tools in representation theory and combinatorics, for example, and include symplectic reflection algebras, rational Cherednik algebras, and Lusztig's graded affine Hecke algebras. These algebras are deformations of the skew group algebra generated by a finite group and a polynomial ring (upon which the group acts). They also provide an algebraic framework for understanding geometric deformations of orbifolds.

Let  $G$  be a finite group acting by linear transformations on a finite dimensional vector space  $V$  over a field  $k$ . Let  $S := S(V)$  be the symmetric algebra with the induced action of  $G$  by automorphisms, and let  $S \# G$  be the corresponding skew group algebra. A graded Hecke algebra (often called a Drinfeld Hecke algebra) emerges after deforming the relations of the symmetric algebra  $S$  inside  $S \# G$ : We

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set each expression  $xy - yx$  (for  $x, y$  in  $V$ ) in the tensor algebra  $T(V)$  equal to an element of the group ring  $kG$  and consider the quotient of  $T(V)\#G$  by such relations. The quotient is a deformation of  $S\#G$  when the relations satisfy certain conditions, and these conditions are explored in many papers (see, e.g., [6, 8, 16, 17]).

Symplectic reflection algebras are special cases of graded Hecke algebras which generalize Weyl algebras in the context of group actions on symplectic spaces. In this paper, we replace Weyl algebras with universal enveloping algebras of Lie algebras and complete the analogy: *Weyl algebras are to symplectic reflection algebras as universal enveloping algebras are to what?* Our answer is the class of Lie orbifold algebras, which together with graded Hecke algebras belong to a larger class of Drinfeld orbifold algebras as we define and explore in this article.

In a previous article [18], we explained that graded Hecke algebras are precisely those deformations of  $S\#G$  which arise from Hochschild 2-cocycles of degree zero with respect to a natural grading on cohomology. In fact, we showed that every such cocycle defines a graded Hecke algebra and thus lifts to a deformation of  $S\#G$ . The present investigation is partly motivated by a desire to understand deformations of  $S\#G$  arising from Hochschild 2-cocycles of degree one.

Specifically, we assign degree 1 to each  $v$  in  $V$  and degree 0 to each  $g$  in  $G$  and consider the corresponding grading on  $T(V)\#G$ . We set each expression  $x\otimes y - y\otimes x$  in the tensor algebra  $T(V)$  equal to an element of degree at most 1 (i.e., nonhomogeneous of filtered degree 1) and consider the quotient of  $T(V)\#G$  by these relations as a filtered algebra. We call the resulting algebra a *Drinfeld orbifold algebra* if it satisfies the Poincaré-Birkhoff-Witt property, i.e., if its associated graded algebra is isomorphic to  $S\#G$ . Such algebras were studied by Halbout, Oudom, and Tang [12] over the real numbers in the special case that  $G$  acts faithfully. We give a direct algebraic approach for arbitrary group actions and fields here.

In this article, we explain in detail the connections between the Poincaré-Birkhoff-Witt property, deformation theory, and Hochschild cohomology. We first classify those deformations of  $S\#G$  which arise as Drinfeld orbifold algebras. We then derive necessary and sufficient conditions on algebra parameters that should facilitate efforts to study and classify these algebras. In particular, we express the PBW property as a set of conditions using the Diamond Lemma. (Although our conditions hold over arbitrary characteristic, we include a comparison with the theory of Koszul rings over  $kG$  used in [8] and [12], which requires  $kG$  to be semisimple.) We give an explicit road map from cohomology, expressed in terms of Koszul resolutions, to the defining relations for Drinfeld orbifold algebras. In particular, we explain how PBW conditions enjoy an elegant description in terms of Gerstenhaber brackets.

Note that one can not automatically deduce results for Drinfeld orbifold algebras defined over  $\mathbb{C}$  from the results in [12] for similar algebras defined over  $\mathbb{R}$ . (For

example, the infinitesimal of a nontrivial deformation over  $\mathbb{C}$  associated to a group  $G$  acting on a complex vector space is always supported off the set  $\mathcal{R}$  of complex reflections in  $G$ , yet the infinitesimal of a nontrivial deformation over  $\mathbb{R}$  associated to that same group (acting on a real vector space of twice the dimension) may have support including  $\mathcal{R}$ .)

More precisely, let us consider a linear “parameter” function mapping the exterior product  $V \wedge V$  to that part of  $T(V)\#G$  having degree at most 1:

$$\kappa : V \wedge V \rightarrow (k \oplus V) \otimes kG .$$

We drop the tensor sign when expressing elements of  $T(V)$  and  $T(V)\#G$  (as is customary when working with noncommutative, associative algebras), writing  $vw$  in place of  $v \otimes w$ , for example. We also usually write  $\kappa(v, w)$  for  $\kappa(v \wedge w)$ , to make some complicated expressions clearer. Define an algebra  $\mathcal{H} := \mathcal{H}_\kappa$  as the quotient

$$\mathcal{H}_\kappa := T(V)\#G / (vw - wv - \kappa(v, w) \mid v, w \in V).$$

We say that  $\mathcal{H}_\kappa$  satisfies the **PBW condition** when its associated graded algebra  $\text{gr } \mathcal{H}_\kappa$  is isomorphic to  $S\#G$  (in analogy with the Poincaré-Birkhoff-Witt Theorem for universal enveloping algebras). In this case, we call  $\mathcal{H}_\kappa$  a **Drinfeld orbifold algebra**. One may check that the PBW condition is equivalent to the existence of a basis  $\{v_1^{m_1} \cdots v_n^{m_n} g : m_i \in \mathbb{Z}_{\geq 0}, g \in G\}$  for  $\mathcal{H}_\kappa$  as a  $k$ -vector space, where  $v_1, \dots, v_n$  is a  $k$ -basis of  $V$ .

The terminology arises because Drinfeld [6] first considered deforming the algebra of coordinate functions  $S^G$  of the orbifold  $V^*/G$  (over  $\mathbb{C}$ ) in this way, although his original construction required the image of  $\kappa$  to lie in the group algebra  $\mathbb{C}G$ . Indeed, when  $\kappa$  has image in  $kG$ , a Drinfeld orbifold algebra  $\mathcal{H}_\kappa$  is called a **Drinfeld Hecke algebra**. These algebras are also called **graded Hecke algebras**, as the graded affine Hecke algebra defined by Lusztig [15, 16] is a special case (arising when  $G$  is a Coxeter group, see [17, Section 3]). Note that symplectic reflection algebras are also examples of these algebras.

Drinfeld orbifold algebras compose a large class of deformations of the skew group algebra  $S\#G$ , as explained in this paper. We determine necessary and sufficient conditions on  $\kappa$  so that  $\mathcal{H}_\kappa$  satisfies the PBW condition and interpret these conditions in terms of Hochschild cohomology. To illustrate, we give several small examples in Sections 3 and 4. We show that a special case of this construction is a class of deformations of the skew group algebras  $\mathcal{U}\#G$ , where  $\mathcal{U}$  is the universal enveloping algebra of a finite dimensional Lie algebra upon which  $G$  acts. These deformations are termed **Lie orbifold algebras**.

For example, consider the Lie algebra  $\mathfrak{sl}_2$  of  $2 \times 2$  matrices over  $\mathbb{C}$  having trace 0 with usual basis  $e, f, h$ . A cyclic group  $G$  of order 2 generated by  $g$  acts as follows:  ${}^g e = f$ ,  ${}^g f = e$ ,  ${}^g h = -h$ . Let  $V$  be the underlying  $\mathbb{C}$ -vector space of  $\mathfrak{sl}_2$  and consider the quotient

$$T(V)\#G / (eh - he + 2e - g, hf - fh + 2f - g, ef - fe - h).$$

We show in Example 4.5 that this quotient is a Lie orbifold algebra. Notice that if we delete the degree 0 term (that is, the group element  $g$ ) in each of the first two relations above, we obtain the skew group algebra  $\mathcal{U}(\mathfrak{sl}_2)\#G$ . If we delete the degree 1 terms instead, we obtain a Drinfeld Hecke algebra (i.e., graded Hecke algebra). (This is a general property of Lie orbifold algebras that we make precise in Proposition 4.2.)

We assume throughout that  $k$  is a field whose characteristic is not 2. For our homological results in Sections 5 through 8, we require in addition that the order of  $G$  is invertible in  $k$  and that  $k$  contains all eigenvalues of the actions of elements of  $G$  on  $V$ ; this assumption is not needed for the first few sections. All tensor products will be over  $k$  unless otherwise indicated.

## 2. DEFORMATIONS OF SKEW GROUP ALGEBRAS

Before exploring necessary and sufficient conditions for an arbitrary quotient algebra to define a Drinfeld orbifold algebra, we explain the connection between these algebras and deformations of the skew group algebra  $S\#G$ . Recall that  $S\#G$  is the  $k$ -vector space  $S \otimes kG$  with algebraic structure given by  $(s_1 \otimes g)(s_2 \otimes h) = s_1 {}^g(s_2) \otimes gh$  for all  $s_i$  in  $S$  and  $g, h$  in  $G$ . Here,  ${}^g s$  denotes the element resulting from the group action of  $g$  on  $s$  in  $S$ . Recall that we drop the tensor symbols and simply write, for example,  $s_1 g s_2 h = s_1 {}^g s_2 g h$ . We show in the next theorem how Drinfeld orbifold algebras arise as a special class of deformations of  $S\#G$ .

First, we recall some standard notation. Let  $R$  be any algebra over the field  $k$ , and let  $t$  be an indeterminate. A **deformation of  $R$  over  $k[t]$**  is an associative  $k[t]$ -algebra with underlying vector space  $R[t]$  and multiplication determined by

$$r * s = rs + \mu_1(r \otimes s)t + \mu_2(r \otimes s)t^2 + \cdots$$

for all  $r, s \in R$ , where  $rs$  is the product of  $r$  and  $s$  in  $R$ , the  $\mu_i : R \otimes R \rightarrow R$  are  $k$ -linear maps that are extended to be  $k[t]$ -linear, and the above sum is finite for each  $r, s$ .

We adapt our definition of  $\mathcal{H}_\kappa$  to that of an algebra over  $k[t]$ . First, decompose  $\kappa$  into its constant and linear parts: Let

$$\kappa = \kappa^C + \kappa^L \quad \text{where} \quad \kappa^C : V \wedge V \rightarrow kG, \quad \kappa^L : V \wedge V \rightarrow V \otimes kG.$$

Write

$$\kappa = \sum_{g \in G} \kappa_g g$$

where each (alternating, bilinear) map  $\kappa_g : V \times V \rightarrow k \oplus V$  also decomposes into constant and linear parts:

$$\kappa_g = \kappa_g^C + \kappa_g^L \quad \text{where} \quad \kappa_g^C : V \wedge V \rightarrow k, \quad \kappa_g^L : V \wedge V \rightarrow V.$$

Now let

$$\mathcal{H}_{\kappa,t} := T(V)\#G[t]/(vw - wv - \kappa^L(v, w)t - \kappa^C(v, w)t^2 \mid v, w \in V).$$

We call  $\mathcal{H}_{\kappa,t}$  a **Drinfeld orbifold algebra over  $k[t]$**  whenever  $\mathcal{H}_{\kappa}$  is a Drinfeld orbifold algebra; in this case,  $\mathcal{H}_{\kappa,t}$  is a deformation of  $S\#G$  over  $k[t]$  and  $\mathcal{H}_{\kappa,t}/t\mathcal{H}_{\kappa,t} \cong S\#G$ .

The following theorem extends [24, Theorem 3.2] (in the case of a trivial twisting cocycle) to our setting. We note that in case  $\kappa^L \equiv 0$ , a change of formal parameter allows us to replace  $t^2$  by  $t$  in the definition of  $\mathcal{H}_{\kappa,t}$ , thus giving the Drinfeld Hecke algebras (i.e., graded Hecke algebras) over  $k[t]$  (defined in [24]) as a special case. We use standard notation for graded linear maps: If  $W$  and  $W'$  are graded vector spaces, a linear map  $\alpha : W \rightarrow W'$  is homogeneous of **degree**  $\deg \alpha$  if  $\alpha(W_i) \subseteq W'_{i+\deg \alpha}$  for all  $i$ .

**Theorem 2.1.** *The Drinfeld orbifold algebras  $\mathcal{H}_{\kappa,t}$  over  $k[t]$  are precisely the deformations of  $S\#G$  over  $k[t]$  for which  $\deg \mu_i = -i$  and for which  $kG$  is in the kernel of  $\mu_i$  for all  $i \geq 1$ .*

The hypothesis that  $kG$  is in the kernel of all  $\mu_i$  is a reasonable one when the characteristic of  $k$  does not divide the order of  $G$ : In this case one may choose to work with maps that are linear over the semisimple ground ring  $kG$  as in [1, 8]. There are however alternative ways to express Drinfeld Hecke algebras for which this hypothesis is not true. See [17, Theorem 3.5] for a comparison with Lusztig's equivalent definition of a Drinfeld (graded) Hecke algebra in which the group action relations are deformed.

*Proof.* Assume  $\mathcal{H}_{\kappa,t}$  is a Drinfeld orbifold algebra over  $k[t]$ . Let  $v_1, \dots, v_n$  be a basis of the vector space  $V$ , so that

$$\{v_1^{i_1} \cdots v_n^{i_n} \mid i_1, \dots, i_n \in \mathbb{Z}_{\geq 0}\}$$

is a basis of  $S$ . Since  $\text{gr}\mathcal{H}_{\kappa} \cong S\#G$ , there is a corresponding basis  $\mathcal{B}$  of  $\mathcal{H}_{\kappa}$  given by all  $v_1^{i_1} \cdots v_n^{i_n} g$ , where  $g$  ranges over all elements in  $G$  and  $i_1, \dots, i_n$  range over all nonnegative integers. Hence, we may identify  $\mathcal{H}_{\kappa,t}$  with  $S\#G[t]$  as a  $k$ -vector space. As  $\mathcal{H}_{\kappa,t}$  is associative, it defines a deformation of  $S\#G[t]$  as follows.

Let  $r = v_1^{i_1} \cdots v_n^{i_n} g$  and  $s = v_1^{j_1} \cdots v_n^{j_n} h$  be elements of  $\mathcal{B}$ . For clarity, we denote the product in  $\mathcal{H}_{\kappa,t}$  by  $*$ . Using the relations of  $\mathcal{H}_{\kappa,t}$  to express the product  $r * s$  as a linear combination of elements of  $\mathcal{B}$ , we may expand uniquely:

$$r * s = rs + \mu_1(r, s)t + \mu_2(r, s)t^2 + \cdots + \mu_m(r, s)t^m$$

for some  $m = m_{r,s}$  depending on  $r, s$ , and some  $\mu_1, \dots, \mu_m$ . By the definition of  $\mathcal{H}_{\kappa,t}$  as a quotient of  $T(V)\#G[t]$ , the group algebra  $kG$  is in the kernel of  $\mu_i$  for all  $i$ . Using the relations in  $\mathcal{H}_{\kappa}$ , we have

$$r * s = ((v_1^{i_1} \cdots v_n^{i_n}) * ({}^g(v_1^{j_1} \cdots v_n^{j_n}))) gh.$$

We apply the relations of  $\mathcal{H}_{\kappa,t}$  repeatedly to rewrite the product  $(v_1^{i_1} \cdots v_n^{i_n}) * ({}^g(v_1^{j_1} \cdots v_n^{j_n}))$  as an element in the  $k$ -span of  $\mathcal{B}$ . We prove by induction on the degree  $d = \sum_{l=1}^n (i_l + j_l)$  that  $\deg \mu_i = -i$ . It suffices to prove this in case  $g = 1$ .

If  $d = 0$  or  $d = 1$ , the maps  $\mu_i$  give 0, and so they satisfy the degree requirement trivially. Similarly, whenever  $a < b$ ,  $v_a * v_b$  in  $\mathcal{H}_{\kappa,t}$  identifies with  $v_a v_b$  in  $S$ , and  $\mu_i(v_a, v_b) = 0$  for all  $i$ . Thus if  $d = 2$ , the nontrivial case is when some  $i_l = 1$  and some  $j_m = 1$  with  $l > m$ . Then

$$v_l * v_m - v_m * v_l = \kappa^L(v_l, v_m)t + \kappa^C(v_l, v_m)t^2.$$

By construction,  $\mu_1(v_l, v_m) = \kappa^L(v_l, v_m)$ , an element of  $V \otimes kG$ , and the map  $\mu_1$  has degree  $-1$  on this input. Similarly,  $\mu_2(v_l, v_m) = \kappa^C(v_l, v_m)$ , which has degree  $-2$  on this input.

Now assume  $d > 2$  is arbitrary and  $d = \sum_{l=1}^n (i_l + j_l)$ . Without loss of generality, assume  $i_n \geq 1$ ,  $j_1 \geq 1$ , and then

$$\begin{aligned} (v_1^{i_1} \cdots v_n^{i_n}) * (v_1^{j_1} \cdots v_n^{j_n}) &= (v_1^{i_1} \cdots v_n^{i_n-1}) * (v_1 v_n * v_1^{j_1-1} \cdots v_n^{j_n}) \\ &\quad + (v_1^{i_1} \cdots v_n^{i_n-1}) * (\kappa^L(v_n, v_1) * v_1^{j_1-1} \cdots v_n^{j_n})t \\ &\quad + (v_1^{i_1} \cdots v_n^{i_n-1}) * (\kappa^C(v_n, v_1) * v_1^{j_1-1} \cdots v_n^{j_n})t^2. \end{aligned}$$

In the second and third terms, we see that the degree lost by applying the map is precisely that gained in the power of  $t$ . In the first term, no degree was lost and no power of  $t$  was gained, however the factors are one step closer to being part of a PBW basis. By induction, the degrees of the  $\mu_i$  are as claimed. Equivalently, we may give  $t$  a degree of 1, making  $\mathcal{H}_{\kappa,t}$  a graded algebra, and argue as in [3, 7].

Now assume that  $A$  is any deformation of  $S \# G$  over  $k[t]$  for which  $\deg \mu_i = -i$  and for which  $kG$  is in the kernel of  $\mu_i$  for all  $i \geq 1$ . By definition,  $A$  is isomorphic to  $S \# G[t]$  as a vector space over  $k[t]$ . Fix a basis  $v_1, \dots, v_n$  of  $V$ . Let  $\phi : T(V) \# G[t] \rightarrow A$  be the  $k[t]$ -linear map given by

$$\phi(v_{i_1} \cdots v_{i_m} g) = v_{i_1} * \cdots * v_{i_m} * g$$

for all words  $v_{i_1} \cdots v_{i_m}$  and group elements  $g$ . Since  $T(V)$  is free on  $v_1, \dots, v_n$  and by hypothesis,  $\mu_i(kG, kG) = \mu_i(kG, V) = \mu_i(V, kG) = 0$  for all  $i \geq 1$ , the map  $\phi$  is in fact an algebra homomorphism. It may be shown by induction on degree that  $\phi$  is surjective, using the degree hypothesis on the maps  $\mu_i$ .

We next find the kernel of  $\phi$ . Let  $v, w \in V$  be elements of the basis. Then

$$\begin{aligned} \phi(vw) &= v * w = vw + \mu_1(v, w)t + \mu_2(v, w)t^2 \\ \phi(wv) &= w * v = wv + \mu_1(w, v)t + \mu_2(w, v)t^2 \end{aligned}$$

since  $\deg \mu_i = -i$  for each  $i$ . Since  $vw = wv$  in  $S$ , we have

$$\phi(vw - wv) = (\mu_1(v, w) - \mu_1(w, v))t - (\mu_2(v, w) - \mu_2(w, v))t^2.$$

It follows that

$$(2.2) \quad vw - wv - (\mu_1(v, w) - \mu_1(w, v))t - (\mu_2(v, w) - \mu_2(w, v))t^2$$

is in the kernel of  $\phi$ , since  $\phi(vg) = vg$  and  $\phi(g) = g$  for all  $v \in V$  and  $g \in G$ . By the degree conditions on the  $\mu_i$ , there are functions  $\kappa_g^L : V \wedge V \rightarrow V \otimes kG$  and

$\kappa_g^C : V \wedge V \rightarrow kG$  for all  $g \in G$  such that

$$(2.3) \quad \mu_1(v, w) - \mu_1(w, v) = \sum_{g \in G} \kappa_g^L(v, w)g$$

$$(2.4) \quad \mu_2(v, w) - \mu_2(w, v) = \sum_{g \in G} \kappa_g^C(v, w)g.$$

For each  $g \in G$ , the functions  $\kappa_g^C : V \wedge V \rightarrow kG$  and  $\kappa_g^L : V \wedge V \rightarrow V \otimes kG$  are linear (by their definitions). Let  $I[t]$  be the ideal of  $T(V) \# G[t]$  generated by all expressions of the form (2.2), so by definition  $I[t] \subset \text{Ker } \phi$ . We claim that in fact  $I[t] = \text{Ker } \phi$ : The quotient  $T(V) \# G[t]/I[t]$  is by definition a filtered algebra over  $k[t]$  whose associated graded algebra is necessarily  $S \# G[t]$  or a quotient thereof. By a dimension count in each degree, since  $I[t] \subset \text{Ker } \phi$ , this forces  $I[t] = \text{Ker } \phi$ . Therefore  $\phi$  induces an isomorphism from  $\mathcal{H}_{\kappa, t}$  to  $A$  and thus the deformation  $A$  of  $S \# G$  is isomorphic to a Drinfeld orbifold algebra.  $\square$

**Remark 2.5.** When working with a Drinfeld orbifold algebra, we may always assume the relations (2.3) and (2.4) hold for  $v, w$  in  $V$  as a consequence of the proof. In a later section, we will make more explicit this connection between the functions  $\mu_i$  and  $\kappa$ , using Hochschild cohomology in case the characteristic of  $k$  does not divide the order of  $G$ : We will consider the  $\mu_i$  to be cochains on the bar resolution of  $S \# G$ , and  $\kappa$  to be a cochain on the Koszul resolution of  $S$ . The relations (2.3) and (2.4) then result from applying chain maps to convert between the two resolutions. Specifically, let  $\phi_\bullet$  be a map from the Koszul resolution to the bar resolution of  $S$  (a subcomplex of the bar resolution of  $S \# G$ ). Then  $\kappa^L = \mu_1 \circ \phi_2$ , as we will explain.

### 3. NECESSARY AND SUFFICIENT CONDITIONS

We determine conditions on the parameter  $\kappa$  for  $\mathcal{H} = \mathcal{H}_\kappa$  to satisfy the PBW condition. In the setting of symplectic reflection algebras, Etingof and Ginzburg [8, Theorem 1.3] used a generalization of results of Braverman and Gaitsgory [3, Theorem 0.5 and Lemma 3.3] that replaces the ground field  $k$  with the (semisimple) group ring  $kG$ . This approach was then adopted in Halbout, Oudom, and Tang [12]. Since one of the conditions in [12] is missing a factor of 2, we include two proofs of the PBW conditions for Drinfeld orbifold algebras, one using this generalization of work of Braverman and Gaitsgory, and one using Bergman's Diamond Lemma [2]. The second proof applies in all characteristics other than 2, even those dividing the order of  $G$ , while the first requires  $kG$  to be semisimple. (See [13] for the Diamond Lemma argument applied in a related setting and see [14] for a related approach using noncommutative Gröbner theory, but in a quantum setting.)

The set of all parameter functions

$$\kappa : V \wedge V \rightarrow (k \oplus V) \otimes kG$$

(defining the quotient algebras  $\mathcal{H}_\kappa$ ) carries the usual induced  $G$ -action:  $({}^h\kappa)(*) = {}^h(\kappa({}^{h^{-1}}(*)))$ , i.e., for all  $h \in G$  and  $v, w \in V$ ,

$$({}^h\kappa)(v, w) = {}^h(\kappa({}^{h^{-1}}v, {}^{h^{-1}}w)) = \sum_{g \in G} {}^h(\kappa_g({}^{h^{-1}}v, {}^{h^{-1}}w)) hgh^{-1}.$$

We say that  $\kappa$  is  $G$ -invariant when  ${}^h\kappa = \kappa$  for all  $h$  in  $G$ . Let  $\text{Alt}_3$  denote the cyclic group of order 3 considered as a subgroup of the symmetric group on 3 symbols. Note that the following theorem gives conditions in the symmetric algebra  $S$ . We allow the field  $k$  to have arbitrary characteristic other than 2.

**Theorem 3.1.** *The algebra  $\mathcal{H}_\kappa$  is a Drinfeld orbifold algebra if and only if the following conditions hold for each  $g$  in  $G$  and  $v_1, v_2, v_3$  in  $V$ :*

- (i) *The parameter function  $\kappa$  is  $G$ -invariant,*
- (ii)  $\sum_{\sigma \in \text{Alt}_3} \kappa_g^L(v_{\sigma(2)}, v_{\sigma(3)})(v_{\sigma(1)} - {}^gv_{\sigma(1)}) = 0$  in  $S = S(V)$ ,
- (iii)  $\sum_{\sigma \in \text{Alt}_3} \sum_{h \in G} \kappa_{gh^{-1}}^L(v_{\sigma(1)} + {}^hv_{\sigma(1)}, \kappa_h^L(v_{\sigma(2)}, v_{\sigma(3)}))$   
 $= 2 \sum_{\sigma \in \text{Alt}_3} \kappa_g^C(v_{\sigma(2)}, v_{\sigma(3)})({}^gv_{\sigma(1)} - v_{\sigma(1)})$ ,
- (iv)  $\sum_{\sigma \in \text{Alt}_3} \sum_{h \in G} \kappa_{gh^{-1}}^C(v_{\sigma(1)} + {}^hv_{\sigma(1)}, \kappa_h^L(v_{\sigma(2)}, v_{\sigma(3)})) = 0$ .

*Proof of Theorem 3.1 using the theory of Koszul rings over  $kG$ .* In this proof, we restrict to the case where the characteristic of  $k$  does not divide the order of  $G$ . The skew group algebra  $S\#G$  is then a Koszul ring over  $kG$ , as defined by Beilinson, Ginzburg, and Soergel (see [1, Definition 1.1.2 and Section 2.6]). These authors worked with graded algebras in which the degree 0 component is not necessarily commutative, but is a semisimple algebra. In our case the degree 0 component of  $S\#G$  is the semisimple group algebra  $kG$ . The results of Braverman and Gaitsgory [3, Theorem 0.5 and Lemma 3.3] can be extended to this general setting to give necessary and sufficient conditions on  $\kappa$  under which  $\mathcal{H}_\kappa$  is a Drinfeld orbifold algebra (cf. [8, Proof of Theorem 1.3]). We give  $V \otimes kG$  a  $kG$ -bimodule structure as follows:  $g(v \otimes h) := {}^gv \otimes gh$  and  $(v \otimes h)g := v \otimes hg$  for all  $v \in V$ ,  $g, h \in G$ . Note that  $T := T_{kG}(V \otimes kG)$  is isomorphic to  $T(V)\#G$  as an algebra.

Let  $P$  be the sub- $kG$ -bimodule of  $T$  generated by all  $v \otimes w - w \otimes v - \kappa(v, w)$ , for  $v, w \in V$ . Let  $R$  be the sub- $kG$ -bimodule generated by all  $v \otimes w - w \otimes v$ , for  $v, w \in V$ . Let  $F$  be the standard filtration on  $T$ , that is,  $F^0(T) = kG$ ,



$F^1(T) = kG \oplus (V \otimes kG)$ ,  $F^2(T) = kG \oplus (V \otimes kG) \oplus (V \otimes V \otimes kG)$ , and so on. By [3, Theorem 0.5],  $\mathcal{H}_\kappa \cong T_{kG}(V \otimes kG)/P$  is a Drinfeld orbifold algebra if and only if

- (I)  $P \cap F^1(T) = 0$  and
- (J)  $(F^1(T) \cdot P \cdot F^1(T)) \cap F^2(T) = P$ .

By [3, Lemma 3.3], if (I) holds, then (J) is equivalent to the following three conditions, where  $\alpha : R \rightarrow V \otimes kG$ ,  $\beta : R \rightarrow kG$  are maps for which

$$P = \{r - \alpha(r) - \beta(r) \mid r \in R\} :$$

- (a)  $\text{Im}(\alpha \otimes \text{id} - \text{id} \otimes \alpha) \subset R$ .
- (b)  $\alpha \circ (\alpha \otimes \text{id} - \text{id} \otimes \alpha) = -(\beta \otimes \text{id} - \text{id} \otimes \beta)$
- (c)  $\beta \circ (\text{id} \otimes \alpha - \alpha \otimes \text{id}) \equiv 0$ .

The above maps  $\alpha \otimes \text{id} - \text{id} \otimes \alpha$  and  $\beta \otimes \text{id} - \text{id} \otimes \beta$  are defined on the intersection  $(R \otimes_{kG} (V \otimes kG)) \cap ((V \otimes kG) \otimes_{kG} R)$ . Extend  $\kappa$  to an alternating  $kG$ -module map on  $T^2 := T_{kG}^2(V \otimes kG)$ , so that  $\kappa(g(v \otimes w)h) = \kappa({}^g v \otimes {}^g w)gh$  for all  $g, h$  in  $G$  and  $v, w$  in  $V$ . Then  $\alpha(v \otimes w - w \otimes v) = \kappa^L(v \otimes w) = (1/2)\kappa^L(v \otimes w - w \otimes v)$  (as  $\kappa$  is alternating) and for all  $r$  in  $R$ ,

$$2\alpha(r) = \kappa^L(r) .$$

(Similarly,  $2\beta(r) = \kappa^C(r)$  for all  $r$  in  $R$ .)

First note that (I) is equivalent to the condition that  $G$  preserves the vector space generated by all  $v \otimes w - w \otimes v - \kappa^L(v, w) - \kappa^C(v, w)$ , i.e., this space contains

$${}^g v \otimes {}^g w - {}^g w \otimes {}^g v - {}^g(\kappa^L(v, w)) - {}^g(\kappa^C(v, w))$$

for each  $g \in G$ ,  $v, w \in V$ . Equivalently,  $\kappa^L({}^g v, {}^g w) = {}^g(\kappa^L(v, w))$  and  $\kappa^C({}^g v, {}^g w) = {}^g(\kappa^C(v, w))$ , i.e., both  $\kappa^L$  and  $\kappa^C$  are  $G$ -invariant, yielding Condition (i) of the theorem.

We assume now that  $\kappa$  is  $G$ -invariant and proceed with the remaining conditions.

Condition (a): As a  $kG$ -bimodule,  $(R \otimes_{kG} (V \otimes kG)) \cap ((V \otimes kG) \otimes_{kG} R)$  is generated by elements of the form  $\sum_{\sigma \in S_3} (\text{sgn } \sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}$ , so we find the image of  $\alpha \otimes \text{id} - \text{id} \otimes \alpha$  on these elements. After reindexing, we obtain

$$\sum_{\sigma \in \text{Alt}_3} (\kappa^L(v_{\sigma(2)}, v_{\sigma(3)}) \otimes v_{\sigma(1)} - v_{\sigma(1)} \otimes \kappa^L(v_{\sigma(2)}, v_{\sigma(3)})) .$$

We decompose into components indexed by  $g$  in  $G$  and shift all group elements to the right (tensor products are over  $kG$ ). The  $g$ -th summand is then

$$(3.2) \quad \sum_{\sigma \in \text{Alt}_3} (\kappa_g^L(v_{\sigma(2)}, v_{\sigma(3)}) \otimes {}^g v_{\sigma(1)} - v_{\sigma(1)} \otimes \kappa_g^L(v_{\sigma(2)}, v_{\sigma(3)}))g ,$$

which must be an element of  $R$ . This is equivalent to the vanishing of its image in  $S \# G$ . We rewrite this as Condition (ii) of the theorem.

Condition (b): We assume Condition (a) holds and thus (3.2) is an element of  $R$ . We compute the left side of Condition (b) by applying  $\alpha$  to this element. Since  $2\alpha(r) = \kappa^L(r)$  for all  $r$  in  $R$ , we obtain the left side of Condition (iii) of the theorem after dividing by 2. Similarly, it is not difficult to see that the right side of Condition (b) agrees with the right side of Condition (iii) of the theorem: The image of  $\sum_{\sigma \in S_3} (\text{sgn } \sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}$  under  $-(\beta \otimes \text{id} - \text{id} \otimes \beta)$  is

$$- \sum_{\sigma \in \text{Alt}_3, g \in G} (\kappa_g^C(v_{\sigma(2)}, v_{\sigma(3)}) \otimes {}^g v_{\sigma(1)} - v_{\sigma(1)} \otimes \kappa_g^C(v_{\sigma(2)}, v_{\sigma(3)}))g$$

(as an element of  $V \otimes kG$ ) which we rewrite as

$$- \sum_{\sigma \in \text{Alt}_3, g \in G} (\kappa_g^C(v_{\sigma(2)}, v_{\sigma(3)})({}^g v_{\sigma(1)} - v_{\sigma(1)}))g.$$

Condition (c): An analysis similar to that for Condition (b) yields Condition (iv) of the theorem.  $\square$

*Proof of Theorem 3.1 using the Diamond Lemma.* In this proof, the characteristic of  $k$  may be 0 or any odd prime. We apply [2] to obtain conditions on  $\kappa$  equivalent to existence of a PBW basis and then argue that these conditions are equivalent to those in the theorem. We suppress details and merely record highlights of the argument (which requires one to fix a monomial ordering and check all overlap/inclusion ambiguities on the set of relations defining  $\mathcal{H}_\kappa$ ), as described, for example, in [4, Chapter 3]). Fix a basis  $v_1, \dots, v_n$  of  $V$  and let  $\mathcal{B}$  be our prospective PBW basis: Set  $\mathcal{B} = \{v_1^{\alpha_1} \cdots v_n^{\alpha_n} g : \alpha_i \in \mathbb{Z}_{\geq 0}, g \in G\} \subset T(V) \otimes kG$ , a subset of the free algebra  $\mathcal{F}$  generated by  $v$  in  $V$  and  $g$  in  $G$ .

Using the Diamond Lemma, one may show that necessary and sufficient conditions for  $\mathcal{H}_\kappa$  to satisfy the PBW condition arise from expanding conjugation and Jacobi identities in  $\mathcal{H}_\kappa$ : For every choice of parameter  $\kappa$ , and for every  $v, w$  in  $V$  and  $h$  in  $G$ , the elements

$$\begin{aligned} (1) \quad & h[v, w]_{\mathcal{H}} h^{-1} - [{}^h v, {}^h w]_{\mathcal{H}}, \quad \text{and} \\ (2) \quad & [v_i, [v_j, v_k]_{\mathcal{H}}]_{\mathcal{H}} + [v_j, [v_k, v_i]_{\mathcal{H}}]_{\mathcal{H}} + [v_k, [v_i, v_j]_{\mathcal{H}}]_{\mathcal{H}} \end{aligned}$$

are always zero in the associative algebra  $\mathcal{H}$ . Here,  $[a, b]_{\mathcal{H}} := ab - ba$  is just the commutator in  $\mathcal{H}$  of  $a, b \in \mathcal{H}$ . Using the relations defining  $\mathcal{H}$ , we move all group elements to the right and arrange indices of basis vectors in increasing order (apply straightening operations).

An analysis of elements of type (1) shows that a PBW property on  $\mathcal{H}_\kappa$  forces  ${}^h \kappa = \kappa$  for all  $h$  in  $G$ . Indeed, this condition is equivalent to

$$\kappa_{h^{-1}gh}(v, w) = {}^{h^{-1}}(\kappa_g({}^h v, {}^h w)) \quad \text{for all } g, h \in G, v, w \in V.$$

We next write each element of type (2) above in the image under the projection map  $\pi : \mathcal{F} \rightarrow \mathcal{H}$  of some  $f(v_i, v_j, v_k)$  in the  $k$ -span of (potentially) nonzero elements

of  $\mathcal{B}$ . In  $\mathcal{H}_\kappa$ ,

$$\begin{aligned}
[v_1, [v_2, v_3]_{\mathcal{H}}]_{\mathcal{H}} &= v_1 \kappa(v_2, v_3) - \kappa(v_2, v_3) v_1 \\
&= \sum_{g \in G} v_1 \kappa_g^C(v_2, v_3) g - \kappa_g^C(v_2, v_3) g v_1 + v_1 \kappa_g^L(v_2, v_3) g - \kappa_g^L(v_2, v_3) g v_1 \\
&= \sum_{g \in G} \left( v_1 \kappa_g^C(v_2, v_3) - \kappa_g^C(v_2, v_3) {}^g v_1 + v_1 \kappa_g^L(v_2, v_3) - \kappa_g^L(v_2, v_3) {}^g v_1 \right) g \\
&= \sum_{g \in G} \left( \kappa_g^C(v_2, v_3) (v_1 - {}^g v_1) + v_1 \kappa_g^L(v_2, v_3) - \kappa_g^L(v_2, v_3) {}^g v_1 \right) g .
\end{aligned}$$

We apply further relations in  $\mathcal{H}$  to this last expression to rearrange the vectors  $v_1, \dots, v_n$  by adding terms of lower degree. Thus, if we express  $f$  as  $f_0 + f_1 + f_2$  where  $f_i$  has degree  $i$  in the free algebra  $\mathcal{F}$ , then  $\pi(f_2)$  and

$$\sum_{g \in G} \sum_{\sigma \in \text{Alt}_3} \kappa_g^L(v_{\sigma(2)}, v_{\sigma(3)}) (v_{\sigma(1)} - {}^g v_{\sigma(1)}) g$$

differ only by a rearrangement of vectors: They both project to the same element under  $T(V) \otimes kG \rightarrow S(V) \otimes kG$ . But  $f_2$  is zero in the free algebra  $\mathcal{F}$  if and only if its image is zero in  $S(V) \otimes kG$ , yielding Condition (ii) of the theorem.

The other conditions of the theorem require a bit of manipulation. One may show that

$$f = \sum_{\substack{\sigma \in \text{Alt}_3 \\ g \in G}} \left[ \kappa_g^C(v_{\sigma(2)}, v_{\sigma(3)}) (v_{\sigma(1)} - {}^g v_{\sigma(1)}) + \sum_{a < b} (D_{ab}^g + D_{ba}^g) v_a v_b - D_{ab}^g \kappa(v_a, v_b) \right] g$$

where the  $D_{ab}^g$  in  $k$  are constants determined by the action of  $G$  on  $V$  and the values of  $\kappa$  expanded in terms of the fixed basis of  $V$ . Specifically,  $D_{ab}^g := \delta_{b, \sigma(1)} c_a^{\sigma(2), \sigma(3), g} - c_b^{\sigma(2), \sigma(3), g} d_a^{\sigma(1), g}$  where  $\delta_{a,b}$  is the Kronecker delta symbol and where  ${}^g v_a = \sum_b d_b^{a,g} v_b$  and  $\kappa_g^L(v_a, v_b) = \sum_m c_m^{a,b,g} v_m$ .

Note that  $f_2$  is zero if and only if  $\sum_{\sigma \in \text{Alt}_3} (D_{ab}^g + D_{ba}^g) = 0$  for all  $a < b$  and  $g$  in  $G$ . Thus, whenever  $f_2$  is zero, we may substitute  $D_{ab}^g = -D_{ba}^g$  in the equation  $0 = 2(f_0 + f_1)$  to see that  $f_0 + f_1$  vanishes exactly when

$$2 \sum_{g \in G} \kappa_g^C(v_{\sigma(2)}, v_{\sigma(3)}) (v_{\sigma(1)} - {}^g v_{\sigma(1)}) g = \sum_{\substack{\sigma \in \text{Alt}_3 \\ g \in G}} \sum_{a < b} (D_{ab}^g - D_{ba}^g) \kappa(v_a, v_b) g .$$

We write the right-hand side as a sum over all  $a$  and  $b$  (as  $\kappa$  is alternating) and obtain

$$\sum_{\substack{\sigma \in \text{Alt}_3 \\ g \in G}} \kappa \left( \kappa_g^L(v_{\sigma(2)}, v_{\sigma(3)}), v_{\sigma(1)} + {}^g v_{\sigma(1)} \right) g .$$

This yields Conditions (iii) and (iv) of the theorem whenever Condition (ii) holds.

Thus, the four conditions of the theorem are equivalent to  $G$ -invariance of  $\kappa$  and the vanishing of all  $f_0, f_1, f_2$  (for any  $i, j, k$ ), which in turn is equivalent to the PBW property for  $\mathcal{H}_\kappa$  by careful application of the Diamond Lemma.  $\square$

We illustrate the theorem by giving two examples for which  $\kappa^C$  is identically 0. In the next section we give an example for which  $\kappa^L$  and  $\kappa^C$  are both nonzero.

**Example 3.3.** Let  $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , with generators  $g$  and  $h$ , act on the complex vector space  $V$  having basis  $x, y, z$  by:

$$\begin{aligned} {}^g x &= -x, & {}^g y &= y, & {}^g z &= -z, \\ {}^h x &= -x, & {}^h y &= -y, & {}^h z &= z. \end{aligned}$$

Define an alternating bilinear map  $\kappa^L : V \times V \rightarrow V \otimes kG$  by

$$\kappa^L(x, y) = zh, \quad \kappa^L(y, z) = xgh, \quad \kappa^L(z, x) = yg,$$

and let  $\kappa^C \equiv 0$ . One may check that  $\kappa^L$  is  $G$ -invariant and that Conditions (ii) and (iii) of Theorem 3.1 hold. Condition (iv) holds automatically since  $\kappa^C$  is identically 0. The corresponding Drinfeld orbifold algebra is

$$T(V) \# G / ([x, y] - zh, [y, z] - xgh, [z, x] - yg).$$

**Example 3.4.** Let  $G = S_3$  act by permutations on a basis  $v_1, v_2, v_3$  of a complex three-dimensional vector space  $V$ . Let  $\xi$  be a primitive cube root of 1, and let

$$w_1 = v_1 + \xi v_2 + \xi^2 v_3, \quad w_2 = v_1 + \xi^2 v_2 + \xi v_3, \quad w_3 = v_1 + v_2 + v_3.$$

Define an alternating bilinear map  $\kappa^L : V \times V \rightarrow V \otimes kG$  by

$$\kappa^L(w_1, w_2) = w_3((1, 2, 3) - (1, 3, 2)), \quad \kappa^L(w_2, w_3) = 0, \quad \kappa^L(w_1, w_3) = 0,$$

and let  $\kappa^C \equiv 0$ , where  $(1, 2, 3), (1, 3, 2)$  are the standard 3-cycles in  $S_3$ . One may check that  $\kappa^L$  is  $G$ -invariant and that Conditions (ii) and (iii) of Theorem 3.1 hold. Condition (iv) holds automatically since  $\kappa^C$  is identically zero. The corresponding Drinfeld orbifold algebra is

$$T(V) \# G / ([w_1, w_2] - w_3((1, 2, 3) - (1, 3, 2)), [w_2, w_3], [w_1, w_3]).$$

The conditions of Theorem 3.1 simplify significantly when  $\kappa^L$  is supported on the identity element  $1 := 1_G$  of  $G$  alone, and we turn to this interesting case in the next section.

## 4. LIE ORBIFOLD ALGEBRAS

The universal enveloping algebra of a finite-dimensional Lie algebra is a special case of a Drinfeld orbifold algebra. We extend universal enveloping algebras by groups and explore deformations of the resulting algebras in this section. Assume throughout this section that the linear part of our parameter  $\kappa$  is supported on the identity  $1 = 1_G$  of  $G$  alone, that is,  $\kappa_g^L \equiv 0$  for all  $g \in G - \{1\}$ . It is convenient in this section to use standard notation from the theory of Lie algebras and Drinfeld Hecke algebras (i.e., graded Hecke algebras): Let

$$a_g : V \wedge V \rightarrow k \quad (\text{for all } g \text{ in } G)$$

and

$$[\cdot, \cdot]_{\mathfrak{g}} : V \wedge V \rightarrow V$$

be linear functions where  $\mathfrak{g} := V$  as a vector space with the additional structure given by the map  $[\cdot, \cdot]_{\mathfrak{g}}$ . Define an algebra  $\mathcal{H} := \mathcal{H}(\mathfrak{g}; a_g, g \in G)$  as the quotient

$$\mathcal{H} = T(V) \# G / (vw - wv - [v, w]_{\mathfrak{g}} - \sum_{g \in G} a_g(v, w)g \mid v, w \in V).$$

Then  $\mathcal{H}$  is a filtered algebra by its definition. We say that  $\mathcal{H}$  is a **Lie orbifold algebra** when it satisfies the PBW condition, that is, when  $\text{gr}\mathcal{H} \cong S \# G$ . We determine necessary and sufficient conditions on the functions  $[\cdot, \cdot]_{\mathfrak{g}}$  and  $a_g$  for  $\mathcal{H}$  to be a Lie orbifold algebra. We will see that the PBW condition implies that  $[\cdot, \cdot]_{\mathfrak{g}}$  defines a Lie bracket on  $V$ , thus explaining the choice of notation and terminology.

We first examine the Jacobi condition of Theorem 3.1 resulting from the Jacobi identity on  $\mathcal{H}$ ,

$$0 = [v_1, [v_2, v_3]_{\mathcal{H}}]_{\mathcal{H}} + [v_2, [v_3, v_1]_{\mathcal{H}}]_{\mathcal{H}} + [v_3, [v_1, v_2]_{\mathcal{H}}]_{\mathcal{H}} \quad \text{for all } v_i \in V,$$

after taking  $\kappa^L(v, w) = [v, w]_{\mathfrak{g}}$  and  $\kappa^C(v, w) = \sum_{g \in G} a_g(v, w)g$ :

**Lemma 4.1.** *Conditions (ii), (iii), and (iv) of Theorem 3.1 (i.e., the Jacobi condition) hold for*

$$\mathcal{H} = T(V) \# G / (v \otimes w - w \otimes v - [v, w]_{\mathfrak{g}} - \sum_{g \in G} a_g(v, w)g \mid v, w \in V)$$

if and only if three conditions are met:

1.  $[\cdot, \cdot]_{\mathfrak{g}}$  endows  $\mathfrak{g} := V$  with the structure of a Lie algebra.
2. The Jacobi identity for Drinfeld Hecke algebras holds: for all  $v_i$  in  $V$ ,  $g$  in  $G$ ,

$$0 = a_g(v_2, v_3)(v_1 - {}^g v_1) + a_g(v_3, v_1)(v_2 - {}^g v_2) + a_g(v_1, v_2)(v_3 - {}^g v_3).$$

3. The constant part of the parameter is compatible with the Lie bracket:

$$0 = a_g(v_3, [v_1, v_2]_{\mathfrak{g}}) + a_g(v_1, [v_2, v_3]_{\mathfrak{g}}) + a_g(v_2, [v_3, v_1]_{\mathfrak{g}}) \quad \text{for all } v_i \text{ in } V, g \text{ in } G.$$

*Proof.* When  $g = 1$ , Theorem 3.1 (ii) holds automatically, and (iii) is equivalent to the Jacobi identity on  $[\cdot, \cdot]_{\mathfrak{g}}$ . For  $g \neq 1$ , Theorem 3.1 (iii) becomes

$$0 = \sum_{\sigma \in \text{Alt}_3} a_g(v_{\sigma(2)}, v_{\sigma(3)}) ({}^g v_{\sigma(1)} - v_{\sigma(1)}),$$

which is Condition (2) in the statement of the lemma. Finally, Theorem 3.1 (iv) becomes Condition (3) in this lemma.  $\square$

Thus when  $\mathcal{H}$  satisfies the PBW condition, we may view the vector space  $V$  as a Lie algebra  $\mathfrak{g}$  under a Lie bracket  $[\cdot, \cdot]_{\mathfrak{g}}$ . We now are ready to give conditions for  $\mathcal{H}$  to be a Lie orbifold algebra in terms of Drinfeld Hecke algebras. The conditions on the bracket  $[\cdot, \cdot]_{\mathfrak{g}}$  stated in the following proposition are immediate consequences of Theorem 3.1(i) and Lemma 4.1(1). We use the notion of “compatible” structures defining a Lie bracket and Drinfeld Hecke algebra as defined in Lemma 4.1(3).

**Proposition 4.2.** *The quotient*

$$\mathcal{H} = T(V) \# G / (v \otimes w - w \otimes v - [v, w]_{\mathfrak{g}} - \sum_{g \in G} a_g(v, w)g \mid v, w \in V)$$

*defines a Lie orbifold algebra if and only if*

1. *The bracket  $[\cdot, \cdot]_{\mathfrak{g}}$  endows  $\mathfrak{g} := V$  with the structure of a Lie algebra upon which  $G$  acts as automorphisms,*
2. *The parameters  $\{a_g\}_{g \in G}$  define a Drinfeld Hecke algebra, and*
3. *The Lie bracket structure is compatible with the Drinfeld Hecke algebra structure.*

In the nonmodular setting, we may use previous analysis of Drinfeld Hecke algebras (i.e., graded Hecke algebras) to state the last proposition in more detail. The conditions on the functions  $a_g$  in the next proposition result from a comparison of Lemma 4.1(2) and the invariance condition of Theorem 3.1(i) with [17, Lemma 1.5, equations (1.6) and (1.7), and Theorem 1.9].

**Proposition 4.3.** *Suppose that  $\text{char}(k)$  does not divide  $|G|$ . Then  $\mathcal{H}$  is a Lie orbifold algebra if and only if*

1. *The map  $[\cdot, \cdot]_{\mathfrak{g}}$  is a  $G$ -invariant Lie bracket,*
- 2a. *The parameters  $\{a_g\}_{g \in G}$  are determined on conjugacy classes, with*

$$a_{hgh^{-1}}(v, w) = a_g({}^h v, {}^h w) \quad \text{for all } v, w \in V, g, h \in G,$$
- 2b. *For each  $g \neq 1$ , either  $a_g \equiv 0$  or  $\text{Ker } a_g = V^g$  with  $\text{codim}(V^g) = 2$ ,*
3. *Each  $a_g$  is compatible with the Lie bracket: for all  $v_i \in V$  and  $g \in G$ ,*

$$0 = a_g(v_3, [v_1, v_2]_{\mathfrak{g}}) + a_g(v_1, [v_2, v_3]_{\mathfrak{g}}) + a_g(v_2, [v_3, v_1]_{\mathfrak{g}}).$$

We further interpret the restrictive Condition (3) in Proposition 4.3: Fix  $g$  in  $G$  with  $\text{codim } V^g = 2$ . Choose vectors  $v_1, v_2$  spanning  $(V^g)^\perp$  and  $v_3, \dots, v_n$  spanning  $V^g$ . This condition then tells us that after expanding with respect to the basis  $v_1, \dots, v_n$ , the coefficient of  $v_2$  in  $[v_2, v_i]_{\mathfrak{g}}$  is equal to the coefficient of  $v_1$  in  $[v_i, v_1]_{\mathfrak{g}}$  for all  $i \geq 3$ .

**Remark 4.4.** We view Lie orbifold algebras as generalizations both of symplectic reflection algebras and of universal enveloping algebras with group actions. Indeed, when  $\mathcal{H}$  is a Lie orbifold algebra, we can replace each function  $a_g$  with the zero function and recover the skew group algebra  $\mathcal{U} \# G$  for  $G$  acting as automorphisms on  $\mathcal{U}$ , the universal enveloping algebra of a finite dimensional Lie algebra. Alternatively, we can replace the linear parameter by zero, i.e., replace  $[\cdot, \cdot]_{\mathfrak{g}}$  by the zero bracket, and recover a Drinfeld Hecke algebra (a symplectic reflection algebra in the special case that  $G$  acts symplectically). Thus, Lie orbifold algebras also include Drinfeld Hecke algebras (and Lusztig's graded affine Hecke algebra, in particular) as special cases. We illustrate by giving details for the example mentioned in the introduction.

**Example 4.5.** Let  $\mathfrak{g} = \mathfrak{sl}_2$  over  $\mathbb{C}$  with basis  $e, f, h$  and Lie bracket defined by

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Let  $G$  be a cyclic group of order 2 generated by  $g$  acting on  $\mathfrak{sl}_2$  by

$${}^g e = f, \quad {}^g f = e, \quad {}^g h = -h.$$

The bracket is  $G$ -invariant under this action. Let  $a_g$  be the skew-symmetric form on  $V = \mathfrak{sl}_2$  defined by

$$a_g(e, h) = 1, \quad a_g(h, f) = 1, \quad a_g(f, e) = 0.$$

This function is  $G$ -invariant, and  $\text{Ker } a_g = V^g$  is the linear span of  $e + f$ , which has codimension 2 in  $V$ . Furthermore,  $a_g$  is compatible with the Lie bracket, that is, Condition (3) of Lemma 4.1 holds. (It suffices to check this condition for  $v_1 = e$ ,  $v_2 = f$ ,  $v_3 = h$ :

$$a_g(h, [e, f]) + a_g(e, [f, h]) + a_g(f, [h, e]) = 0.)$$

Set  $a_1$  equal to the zero function. Then  $T(V) \# G$  modulo the ideal generated by

$$eh - he + 2e - g, \quad hf - fh + 2f - g, \quad ef - fe - h$$

is a Lie orbifold algebra.

In fact, Theorem 3.1 shows there are only two parameters' worth of Lie orbifold algebras capturing this action of  $G$  on  $\mathfrak{sl}_2$ : Every such Lie orbifold algebra has the form

$$T(V) \# G / (eh - he + 2e - t_2g + t_1, \quad hf - fh + 2f - t_2g + t_1, \quad ef - fe - h)$$

for some scalars  $t_1, t_2$  in  $\mathbb{C}$ . (Note that  $t_1 = t_2 = 0$  defines the universal enveloping algebra extended by  $G$ .)

## 5. KOSZUL RESOLUTION

Hochschild cohomology catalogues and illuminates deformations of an algebra. Indeed, every deformation of a  $k$ -algebra  $R$  corresponds to an element in degree 2 Hochschild cohomology,  $\mathrm{HH}^2(R)$ . Isomorphic deformations define cohomologous cocycles. We isolated in [18] the cocycles that define Drinfeld Hecke algebras (i.e., graded Hecke algebras): Drinfeld Hecke algebras are precisely those deformations of  $S\#G$  whose corresponding Hochschild 2-cocycles are “constant”. In the next section, we explain this statement, and we more generally express conditions for a quotient of  $T(V)\#G$  to define a Drinfeld orbifold algebra in terms of the Hochschild cohomology of  $S\#G$  and its graded Lie structure. Here we establish preliminaries and notation. *From now on, we assume that the characteristic of  $k$  does not divide the order of  $G$  and that  $k$  contains the eigenvalues of the actions of elements of  $G$  on  $V$ .* (For example, take  $k$  algebraically closed of characteristic coprime to  $|G|$ .)

Recall that we denote the image of  $v$  in  $V$  under the action of any  $g$  in  $G$  by  ${}^g v$ . Write  $V^*$  for the contragredient (or dual) representation. Given any basis  $v_1, \dots, v_n$  of  $V$ , let  $v_1^*, \dots, v_n^*$  denote the dual basis of  $V^*$ . Given any set  $A$  carrying an action of  $G$ , we write  $A^G$  for the subset of elements invariant under the action. Again, we write  $V^g$  for the  $g$ -invariant subspace of  $V$ . Since  $G$  is finite, we may assume  $G$  acts by isometries on  $V$  (i.e.,  $G$  preserves a Hermitian form on  $V$ ).

The Hochschild cohomology  $\mathrm{HH}^\bullet(S\#G)$  is the space  $\mathrm{Ext}_{(S\#G)^e}^\bullet(S\#G, S\#G)$ , where  $(S\#G)^e = (S\#G) \otimes (S\#G)^{op}$  acts on  $S\#G$  by multiplication, one tensor factor acting on the left and the other tensor factor acting on the right. We also examine the Hochschild cohomology  $\mathrm{HH}^\bullet(S, S\#G) := \mathrm{Ext}_{S^e}^\bullet(S, S\#G)$  where  $S^e = S \otimes S^{op}$  and, more generally,  $\mathrm{HH}^\bullet(S, M) := \mathrm{Ext}_{S^e}^\bullet(S, M)$  for any  $S^e$ -module  $M$ .

Let  $\mathcal{C}$  be a set of representatives of the conjugacy classes of  $G$ . For any  $g$  in  $G$ , let  $Z(g)$  be the centralizer of  $g$ . Since we have assumed that the characteristic of  $k$  does not divide the order of  $G$ , there is a  $G$ -action giving the first of the following isomorphisms of graded vector spaces (see, for example, Ştefan [22, Cor. 3.4]):

$$\begin{aligned}
 \mathrm{HH}^\bullet(S\#G) &\cong \mathrm{HH}^\bullet(S, S\#G)^G \\
 (5.1) \qquad &\cong \left( \bigoplus_{g \in G} \mathrm{HH}^\bullet(S, Sg) \right)^G \cong \bigoplus_{g \in \mathcal{C}} \mathrm{HH}^\bullet(S, Sg)^{Z(g)}.
 \end{aligned}$$

The first line is in fact a graded algebra isomorphism; it follows from applying a spectral sequence. The second isomorphism results from decomposing the bimodule  $S\#G$  into the direct sum of components  $Sg$ . The action of  $G$  permutes these components via the conjugation action of  $G$  on itself, and thus the third isomorphism is a canonical projection onto a set of representative summands. Each



space  $\mathrm{HH}^\bullet(S, Sg) = \mathrm{Ext}_{S^e}^\bullet(S, Sg)$  may be determined explicitly using the Koszul resolution of  $S$  (a free resolution of  $S$  as an  $S^e$ -module) that we recall next.

The **Koszul resolution**  $K_\bullet(S)$  is defined by  $K_0(S) = S^e$  and

$$(5.2) \quad K_p(S) = S^e \otimes \bigwedge^p(V)$$

for  $p \geq 1$ , with differentials

$$(5.3) \quad d_p(1 \otimes 1 \otimes v_{j_1} \wedge \cdots \wedge v_{j_p}) = \sum_{i=1}^p (-1)^{i+1} (v_{j_i} \otimes 1 - 1 \otimes v_{j_i}) \otimes (v_{j_1} \wedge \cdots \wedge \hat{v}_{j_i} \wedge \cdots \wedge v_{j_p})$$

for all  $v_{j_1}, \dots, v_{j_p} \in V$  (e.g., see Weibel [23, §4.5]). We apply  $\mathrm{Hom}_{S^e}(-, Sg)$  to each term of the Koszul resolution and then identify

$$\mathrm{Hom}_{S^e}(S^e \otimes \bigwedge^p(V), Sg) \cong \mathrm{Hom}_k(\bigwedge^p V, Sg) \cong Sg \otimes \bigwedge^p V^*$$

for each  $g$  in  $G$ . Thus we write the set of **cochains** arising from the Koszul resolution (from which the cohomology classes emerge) as vector forms on  $V$  tagged by group elements: Let

$$(5.4) \quad C^\bullet = \bigoplus_{g \in G} C_g^\bullet, \quad \text{where} \quad C_g^p := Sg \otimes \bigwedge^p V^* \quad \text{for each } g \in G.$$

We call  $C_g^\bullet$  the space of cochains **supported on**  $g$ . Similarly, for any subset  $X$  of  $G$ , we define  $C_X^\bullet := \bigoplus_{g \in X} C_g^\bullet$ , the set of cochains **supported on**  $X$ . We say a cochain in  $C^\bullet$  is **supported off** a subset  $X$  of  $G$  if it lies in  $\bigoplus_{g \notin X} C_g^\bullet$ . Note that each element of  $G$  permutes the summands of  $C^\bullet$  via the conjugation action of  $G$  on itself.

From the space  $C^\bullet$  of cochains, we define a space of representatives of cohomology classes: Let

$$(5.5) \quad H^\bullet := \bigoplus_{g \in G} S(V^g)g \otimes \bigwedge^{\bullet - \mathrm{codim} V^g} (V^g)^* \otimes \bigwedge^{\mathrm{codim} V^g} ((V^g)^\perp)^*.$$

Then  $H^\bullet \subset C^\bullet$  with

$$H^\bullet \cong \mathrm{HH}^\bullet(S, S\#G) \quad \text{and} \quad (H^\bullet)^G \cong \mathrm{HH}^\bullet(S, S\#G)^G \cong \mathrm{HH}^\bullet(S\#G).$$

(See [21, Proposition 5.11 and (6.1)] for this formulation of the Hochschild cohomology. It was computed first independently by Farinati [9] and by Ginzburg and Kaledin [10].) In particular it follows that  $(H^2)^G$  is supported on elements  $g$  for which  $\mathrm{codim} V^g \in \{0, 2\}$ , since an element of  $(H^2)^G$  is invariant under the action of each group element  $g$ . See [18, Lemma 3.6] for details.

The grading on the polynomial ring  $S = S(V)$  induces a grading on the set of cochains by polynomial degree: We say a cochain in  $C^\bullet$  has **polynomial degree**  $i$  if the factors in  $S$  in the expression (5.4) are all polynomials of degree  $i$ . We say a cochain is **homogeneous** when its polynomial factors in  $S$  are homogeneous. A **constant cochain** is then one of polynomial degree 0 and a **linear cochain** is one of homogeneous polynomial degree 1. The cochains  $C^\bullet$  are filtered by polynomial

degree:  $C_0^\bullet \subset C_1^\bullet \subset C_2^\bullet \subset \cdots$ , where  $C_i^\bullet$  is the subspace of  $C^\bullet$  consisting of cochains of polynomial degree at most  $i$ .

**Definition 5.6.** *We define a **cochain bracket map** on the subspace generated by linear and constant 2-cochains: Let  $[\ast, \ast] : C_1^2 \times C_1^2 \rightarrow C_1^3$  be the symmetric map defined by  $[\alpha, \beta](v_1, v_2, v_3)$*

$$:= \begin{cases} \sum_{\substack{g, h \in G \\ \sigma \in \text{Alt}_3}} \left[ \alpha_{gh^{-1}} \left( \beta_h(v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge v_{\sigma(3)}) \right) + \beta_{gh^{-1}} \left( \alpha_h(v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge v_{\sigma(3)}) \right) \right] g & \text{for linear } \alpha, \beta \\ \sum_{\substack{g, h \in G \\ \sigma \in \text{Alt}_3}} \alpha_{gh^{-1}} \left( \beta_h(v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge v_{\sigma(3)}) \right) g & \text{for constant } \alpha \text{ and linear } \beta \\ 0 & \text{for constant } \alpha \text{ and } \beta \end{cases}$$

for all  $v_1, v_2, v_3$  in  $V$ .

We will see in the next section that this definition gives a representative cochain for a class in cohomology  $\text{HH}^\bullet(S \# G)$  of the Gerstenhaber bracket of  $\alpha$  and  $\beta$  when they are cocycles.

## 6. GERSTENHABER BRACKET

In this section we recall the definition of the Gerstenhaber bracket on Hochschild cohomology, defined on the bar resolution, and show how it is related to the cochain bracket map of Definition 5.6. Recall the definition of the **bar resolution** of a  $k$ -algebra  $R$ : It has  $p$ th term  $R^{\otimes(p+2)}$  and differentials

$$\delta_p(r_0 \otimes \cdots \otimes r_{p+1}) = \sum_{i=0}^p (-1)^i r_0 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_{p+1}$$

for all  $r_0, \dots, r_{p+1} \in R$ . From this one may derive the standard definition of a **Hochschild 2-cocycle**: It is an element  $\mu$  of  $\text{Hom}_k(R \otimes R, R) \cong \text{Hom}_{R^e}(R^{\otimes 2}, R)$  for which

$$(6.1) \quad \mu(rs, u) + \mu(r, s)u = \mu(r, su) + r\mu(s, u)$$

for all  $r, s, u \in R$ . (Here we have further identified the linear map  $\mu$  on  $R \otimes R$  with a bilinear map on  $R \times R$ .)

We will need the following lemma for our calculations.

**Lemma 6.2.** *Let  $\mu$  be a Hochschild 2-cocycle on  $S \# G$  whose kernel contains  $kG$ . Then*

$$\mu(rg, s) = \mu(r, gs) = \mu(r, {}^g s)g$$

for all  $r, s$  in  $S$  and  $g$  in  $G$ .

*Proof.* Apply (6.1) to  $r, g, s$  to obtain  $\mu(rg, s) + \mu(r, g)s = \mu(r, gs) + r\mu(g, s)$ . By hypothesis,  $\mu(r, g) = 0 = \mu(g, s)$ , so  $\mu(rg, s) = \mu(r, gs)$ . Now apply (6.1) to  $r, {}^gs, g$  to obtain  $\mu(r({}^gs), g) + \mu(r, {}^gs)g = \mu(r, ({}^gs)g) + r\mu({}^gs, g)$ . By hypothesis,  $\mu(r({}^gs), g) = 0 = \mu({}^gs, g)$ , so  $\mu(r, {}^gs)g = \mu(r, ({}^gs)g)$ . Since  $gs = ({}^gs)g$ , the lemma follows.  $\square$

We will also need the definition of the circle operation on Hochschild cohomology in degree 2: If  $R$  is a  $k$ -algebra and  $\alpha$  and  $\beta$  are elements of  $\text{Hom}_{R^e}(R^{\otimes 4}, R) \cong \text{Hom}_k(R^{\otimes 2}, R)$ , then  $\alpha \circ \beta \in \text{Hom}_k(R^{\otimes 3}, R)$  is defined by

$$\alpha \circ \beta(r_1 \otimes r_2 \otimes r_3) := \alpha(\beta(r_1 \otimes r_2) \otimes r_3) - \alpha(r_1 \otimes \beta(r_2 \otimes r_3))$$

for all  $r_1, r_2, r_3 \in R$ . The **Gerstenhaber bracket** is then

$$[\alpha, \beta] := \alpha \circ \beta + \beta \circ \alpha.$$

This bracket is well-defined on cohomology classes, however the circle operation is not. In our setting,  $R = S\#G$ , and we now express the Gerstenhaber bracket on input from the Koszul resolution using the cochain bracket of Definition 5.6. In the theorem below, we fix a choice of isomorphism  $\text{HH}^\bullet(S\#G) \cong (H^\bullet)^G$  where  $H^\bullet$  is given by (5.5). (See [21, Proposition 5.11 and (6.1)].)

**Theorem 6.3.** *Consider two cohomology classes  $\alpha', \beta'$  in  $\text{HH}^2(S\#G)$  represented by cochains  $\alpha, \beta$  in  $(H^2)^G$  of polynomial degree at most 1. Then the Gerstenhaber bracket in  $\text{HH}^3(S\#G)$  of  $\alpha'$  and  $\beta'$  is represented by the cochain bracket  $[\alpha, \beta]$  of Definition 5.6.*

*Proof.* We use the chain map  $\phi_\bullet$  from the Koszul resolution  $K_\bullet(S)$  to the bar resolution for  $S = S(V)$  given in each degree by

$$(6.4) \quad \phi_p(1 \otimes 1 \otimes v_{j_1} \wedge \cdots \wedge v_{j_p}) = \sum_{\sigma \in \text{Sym}_p} \text{sgn}(\sigma) \otimes v_{j_{\sigma(1)}} \otimes \cdots \otimes v_{j_{\sigma(p)}} \otimes 1$$

for all  $v_{j_1}, \dots, v_{j_p} \in V$ , where  $\text{Sym}_p$  denotes the symmetric group on  $p$  symbols. We may view functions on the bar resolution in cohomological degree 2 as functions on  $K_2(S) = S^e \otimes \bigwedge^2(V)$  simply by composing with  $\phi_2$ .

We will also need a choice  $\psi_\bullet$  of chain map from the bar to the Koszul resolution. The particular choice of  $\psi_\bullet$  does not matter here, but we will assume that  $\psi\phi$  is the identity map and that  $\psi_2(1 \otimes a \otimes b \otimes 1) = 0$  if either  $a$  or  $b$  is in the field  $k$ . (For example, one could take  $\psi_\bullet$  so that  $\psi_2(1 \otimes v_i \otimes v_j \otimes 1) = 1 \otimes 1 \otimes v_i \wedge v_j$  for  $i < j$  and 0 otherwise, for some fixed basis  $v_1, \dots, v_n$  of  $V$ . See [20] for explicit constructions of such maps  $\psi_\bullet$ ; we will not need them here.) Note that although  $\psi_2$  may not be a  $kG$ -homomorphism, the map  $\psi_2$  preserves the action of  $G$  on the image of  $\phi_2$ . For our purposes here, this implies that we do not need to average over  $G$  when computing brackets, as is done, e.g., in [21]. We also note that every chain map  $\psi$  for which  $\psi\phi$  is the identity map has the property that

$$\psi_2(1 \otimes v \otimes w \otimes 1 - 1 \otimes w \otimes v \otimes 1) = 1 \otimes 1 \otimes v \wedge w$$

for all  $v, w$  in  $V$ . Thus on elements of this form,  $\psi_2$  is independent of choice of basis of  $V$ .

We extend each function  $\gamma$  on the bar complex for  $S$  to a function on the bar complex for  $S\#G$  in a standard way: In degree 2, we require  $kG$  to be in the kernel of  $\gamma$  and set

$$\gamma(1 \otimes s_1 g_1 \otimes s_2 g_2 \otimes 1) := \gamma(1 \otimes s_1 \otimes {}^{g_1}s_2 \otimes 1) g_1 g_2$$

for  $\gamma \in \text{Hom}_{S^e}(S^{\otimes 4}, S\#G)$ ,  $s_1, s_2 \in S$ ,  $g_1, g_2 \in G$ . Compare with Lemma 6.2. (See [5, Theorem 5.4] for a more general statement.)

We apply the chain map  $\psi$  to convert  $\alpha$  and  $\beta$  to functions on the bar complex, execute the Gerstenhaber bracket, and apply  $\phi$  to convert back to a function on the Koszul complex. The induced operation on cochains arising from the Koszul complex is thus

$$[\alpha, \beta] := \phi^*(\psi^*(\alpha) \circ \psi^*(\beta)) + \phi^*(\psi^*(\beta) \circ \psi^*(\alpha)) .$$

Note there is no guarantee that  $[\alpha, \beta]$  is in the chosen space  $H^3$  of representatives of cohomology classes, however there is a unique element of  $H^3$  to which it is cohomologous.

We compute separately the two corresponding circle operations, keeping in mind that they are not well-defined on cohomology, and so must be combined. (Again, we identify  $\text{Hom}_{S^e}(S^{\otimes(p+2)}, *)$  with  $\text{Hom}_k(S^{\otimes p}, *)$  and  $\text{Hom}_{S^e}(S^e \otimes \bigwedge^p V, *)$  with  $\text{Hom}_k(\bigwedge^p V, *)$ , dropping extra tensor factors of 1.) Then

$$\begin{aligned} (\alpha \circ \beta)(v_1 \wedge v_2 \wedge v_3) &= (\psi^*(\alpha) \circ \psi^*(\beta))\phi(v_1 \wedge v_2 \wedge v_3) \\ &= (\psi^*(\alpha) \circ \psi^*(\beta)) \sum_{\sigma \in S_3} \text{sgn}(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)} \\ &= \sum_{\sigma \in S_3} \text{sgn}(\sigma) \psi^*(\alpha) \left( \psi^*(\beta)(v_{\sigma(1)} \otimes v_{\sigma(2)}) \otimes v_{\sigma(3)} - v_{\sigma(1)} \otimes \psi^*(\beta)(v_{\sigma(2)} \otimes v_{\sigma(3)}) \right). \end{aligned}$$

We may rewrite the sum over the alternating group instead to obtain

$$\begin{aligned} (\alpha \circ \beta)(v_1 \wedge v_2 \wedge v_3) &= \sum_{\sigma \in \text{Alt}_3} \psi^*(\alpha) \left( \psi^*(\beta)(v_{\sigma(1)} \otimes v_{\sigma(2)}) \otimes v_{\sigma(3)} - \psi^*(\beta)(v_{\sigma(2)} \otimes v_{\sigma(1)}) \otimes v_{\sigma(3)} \right) \\ &\quad - \psi^*(\alpha) \left( v_{\sigma(1)} \otimes \psi^*(\beta)(v_{\sigma(2)} \otimes v_{\sigma(3)}) - v_{\sigma(1)} \otimes \psi^*(\beta)(v_{\sigma(3)} \otimes v_{\sigma(2)}) \right). \end{aligned}$$

But  $\psi^*(\beta)(v \otimes w - w \otimes v) = \beta(v \wedge w)$  for all vectors  $v, w$  in  $V$ , and hence Lemma 6.2 implies that the above sum is just

$$\begin{aligned}
 (6.5) \quad & \sum_{\sigma \in \text{Alt}_3} \psi^*(\alpha) \left( \beta(v_{\sigma(1)} \wedge v_{\sigma(2)}) \otimes v_{\sigma(3)} - v_{\sigma(1)} \otimes \beta(v_{\sigma(2)} \wedge v_{\sigma(3)}) \right) \\
 &= \sum_{\sigma \in \text{Alt}_3, h \in G} \psi^*(\alpha) \left( \beta_h(v_{\sigma(1)} \wedge v_{\sigma(2)}) h \otimes v_{\sigma(3)} - v_{\sigma(1)} \otimes \beta_h(v_{\sigma(2)} \wedge v_{\sigma(3)}) h \right) \\
 &= \sum_{\sigma \in \text{Alt}_3, h \in G} \psi^*(\alpha) \left( \beta_h(v_{\sigma(1)} \wedge v_{\sigma(2)}) \otimes {}^h v_{\sigma(3)} - v_{\sigma(1)} \otimes \beta_h(v_{\sigma(2)} \wedge v_{\sigma(3)}) \right) h.
 \end{aligned}$$

First assume the polynomial degree of  $\beta$  is 0. Then each  $\beta_h(v_i \wedge v_j)$  is constant. But  $\psi^*(\alpha)(a \otimes b)$  is zero for either  $a$  or  $b$  in  $k$ , and the last expression is thus zero. Hence,  $\alpha \circ \beta(v_1 \wedge v_2 \wedge v_3)$  is zero for  $\beta$  of polynomial degree 0.

Now assume  $\beta$  is homogenous of polynomial degree 1. We claim that for any  $h$  in  $G$  and any  $u_1, u_2, u_3$  in  $V$ ,

$$(6.6) \quad \sum_{\sigma \in \text{Alt}_3} \beta_h(u_{\sigma(1)} \wedge u_{\sigma(2)}) \otimes {}^h u_{\sigma(3)} = \sum_{\sigma \in \text{Alt}_3} \beta_h(u_{\sigma(1)} \wedge u_{\sigma(2)}) \otimes u_{\sigma(3)}.$$

The equation clearly holds for  $h$  acting trivially on  $V$ . One may easily verify the equation for  $h$  not in the kernel of the representation  $G \rightarrow \text{GL}(V)$  by fixing a basis of  $V$  consisting of eigenvectors for  $h$  and using the fact that any nonzero  $\beta_h$  is supported on  $\bigwedge^2(V^h)^\perp$  with  $\text{codim } V^h = 2$ ; see (5.5).

We use Equation 6.6 and the fact that  $\psi^*(\alpha)(v \otimes w - w \otimes v) = \alpha(v \wedge w)$  for all vectors  $v, w$  in  $V$  to simplify Equation 6.5:

$$\begin{aligned}
 & (\alpha \circ \beta)(v_1 \wedge v_2 \wedge v_3) \\
 &= \sum_{\sigma \in \text{Alt}_3, h \in G} \psi^*(\alpha) \left( \beta_h(v_{\sigma(1)} \wedge v_{\sigma(2)}) \otimes {}^h v_{\sigma(3)} - v_{\sigma(1)} \otimes \beta_h(v_{\sigma(2)} \wedge v_{\sigma(3)}) \right) h \\
 &= \sum_{\sigma \in \text{Alt}_3, h \in G} \psi^*(\alpha) \left( \beta_h(v_{\sigma(1)} \wedge v_{\sigma(2)}) \otimes v_{\sigma(3)} - v_{\sigma(1)} \otimes \beta_h(v_{\sigma(2)} \wedge v_{\sigma(3)}) \right) h \\
 &= \sum_{\sigma \in \text{Alt}_3, h \in G} \psi^*(\alpha) \left( \beta_h(v_{\sigma(1)} \wedge v_{\sigma(2)}) \otimes v_{\sigma(3)} - v_{\sigma(3)} \otimes \beta_h(v_{\sigma(1)} \wedge v_{\sigma(2)}) \right) h \\
 &= \sum_{\sigma \in \text{Alt}_3, h \in G} \alpha \left( \beta_h(v_{\sigma(1)} \wedge v_{\sigma(2)}) \wedge v_{\sigma(3)} \right) h \\
 &= \sum_{\sigma \in \text{Alt}_3; g, h \in G} \alpha_g \left( \beta_h(v_{\sigma(1)} \wedge v_{\sigma(2)}) \wedge v_{\sigma(3)} \right) gh.
 \end{aligned}$$

A similar computation for  $\beta \circ \alpha$  together with reindexing over the group yields the result.  $\square$

The Gerstenhaber bracket takes a particularly nice form when we consider square brackets of linear cocycles and brackets of linear with constant cocycles:

**Corollary 6.7.** *Consider cohomology classes  $\alpha', \beta'$  in  $\mathrm{HH}^2(S\#G)$  represented respectively by a constant cocycle  $\alpha$  and a linear cocycle  $\beta$  in  $(H^2)^G$ . The Gerstenhaber bracket in  $\mathrm{HH}^3(S\#G)$  of  $\beta'$  with itself and of  $\alpha'$  with  $\beta'$  are represented by the cocycles*

$$[\beta, \beta](v_1, v_2, v_3) = 2 \sum_{\substack{g, h \in G \\ \sigma \in \mathrm{Alt}_3}} \beta_{gh^{-1}} \left( \beta_h(v_{\sigma(1)} \wedge v_{\sigma(2)}) \wedge v_{\sigma(3)} \right) g$$

and

$$[\alpha, \beta](v_1, v_2, v_3) = \sum_{\substack{g, h \in G \\ \sigma \in \mathrm{Alt}_3}} \alpha_{gh^{-1}} \left( \beta_h(v_{\sigma(1)} \wedge v_{\sigma(2)}) \wedge v_{\sigma(3)} \right) g ,$$

respectively.

## 7. PBW CONDITION AND GERSTENHABER BRACKET

In this section, we give necessary and sufficient conditions on a parameter to define a Drinfeld orbifold algebra in terms of Hochschild cohomology. We interpret Theorem 3.1 in terms of cocycles and the Gerstenhaber bracket in cohomology as realized on the set of cochains arising from the Koszul resolution. Our results should be compared with [12, §2.2, (4), (5), (6)], where a factor of 2 is missing from the right side of (5). See also [13, (1.9)] for a somewhat different setting.

We want to describe precisely which parameter maps  $\kappa$  result in a quotient  $\mathcal{H}_\kappa$  that satisfies the PBW condition, that is, defines a Drinfeld orbifold algebra. The algebras  $\mathcal{H}_\kappa$  are naturally expressed and analyzed in terms of the Koszul resolution of  $S$ . Recall,  $\kappa : \bigwedge^2 V \rightarrow S \otimes \mathbb{C}G$  with  $\kappa = \sum_{g \in G} \kappa_g g$ . The parameter map  $\kappa$  as well as its linear and constant parts,  $\kappa^L$  and  $\kappa^C$ , thus define cochains on the Koszul resolution and we identify  $\kappa, \kappa^L, \kappa^C$  with elements of  $C^\bullet$ . Indeed, for each  $g \in G$ , the functions  $\kappa_g g$ ,  $\kappa_g^L g$ , and  $\kappa_g^C g$  (from  $\bigwedge^2 V$  to  $Sg$ ) define elements of the cochain complex  $C_g^\bullet$  of (5.4).

We now determine a complete set of necessary and sufficient conditions on these parameters regarded as cochains in Hochschild cohomology  $\mathrm{HH}^\bullet(S\#G) \cong \mathrm{HH}^\bullet(S, S\#G)^G$ . (We use the chain maps converting between resolutions discussed in Section 6.) The significance of the following lemma and theorem thereafter lies in the expression of the PBW property in terms of the Gerstenhaber bracket in cohomology. We distinguish a cochain  $[\alpha, \beta]$  on the Koszul resolution (5.2) from its cohomology class arising from the induced Gerstenhaber bracket by using the phrase “as a cochain” where appropriate. (Recall that  $d$  is the differential on the Koszul resolution defined as in (5.3).)

**Lemma 7.1.** *In Theorem 3.1,*

- Condition (ii) holds if and only if  $\kappa^L$  is a cocycle, i.e.,  $d^*\kappa^L = 0$ .
- For  $\kappa^L$  in  $H^\bullet$ , Condition (iii) is equivalent to  $[\kappa^L, \kappa^L] = 2d^*\kappa^C$  as cochains.
- For  $\kappa^L$  in  $H^\bullet$ , Condition (iv) is equivalent to  $[\kappa^C, \kappa^L] = 0$  as a cochain.

*Proof.* The cochain  $d^*\kappa^L$  is zero exactly when  $\kappa^L$  takes to 0 all input of the form

$$d_3(v_1 \wedge v_2 \wedge v_3) = (v_1 \otimes 1 - 1 \otimes v_1) \otimes v_2 \wedge v_3 - (v_2 \otimes 1 - 1 \otimes v_2) \otimes v_1 \wedge v_3 \\ + (v_3 \otimes 1 - 1 \otimes v_3) \otimes v_1 \wedge v_2,$$

in other words, when

$$0 = v_1\kappa^L(v_2, v_3) - \kappa^L(v_2, v_3)v_1 + v_2\kappa^L(v_3, v_1) - \kappa^L(v_3, v_1)v_2 + v_3\kappa^L(v_1, v_2) - \kappa^L(v_1, v_2)v_3$$

in  $S\#G$ . This is equivalent to

$$0 = v_1\kappa_g^L(v_2, v_3)g - \kappa_g^L(v_2, v_3)gv_1 + v_2\kappa_g^L(v_3, v_1)g \\ - \kappa_g^L(v_3, v_1)gv_2 + v_3\kappa_g^L(v_1, v_2)g - \kappa_g^L(v_1, v_2)gv_3$$

for each  $g$  in  $G$ . We rewrite this expression using the commutativity of  $S$  and moving all factors of  $g$  to the right:

$$0 = \kappa_g^L(v_2, v_3)(v_1 - {}^g v_1) + \kappa_g^L(v_3, v_1)(v_2 - {}^g v_2) + \kappa_g^L(v_1, v_2)(v_3 - {}^g v_3),$$

which is precisely Theorem 3.1(ii).

Next, notice that we may apply Equation 6.6 (in the proof of Theorem 6.3) to  $\beta = \kappa^L$ , under the assumption that  $\kappa^L$  lies in  $H^2$ . Then for each  $g$  in  $G$ , the left side of Theorem 3.1(iii) is the opposite of the coefficient of  $g$  in  $[\kappa^L, \kappa^L]$  by Definition 5.6 (see Corollary 6.7) and the skew-symmetry of  $\kappa^L$ . By a similar calculation to that for  $\kappa^L$ , the right side of Theorem 3.1(iii) is the coefficient of  $g$  in

$$-2d_3^*\kappa^C(v_1 \wedge v_2 \wedge v_3) = 2 \sum_{g \in G} \sum_{\sigma \in \text{Alt}_3} \kappa_g^C(v_{\sigma(2)}, v_{\sigma(3)})({}^g v_{\sigma(1)} - v_{\sigma(1)})g.$$

Hence, Theorem 3.1(iii) is equivalent to  $[\kappa^L, \kappa^L] = 2d_3^*\kappa^C$ . This condition differs from [12, (5)] where the factor of 2 is missing.

We again compare coefficients of fixed  $g$  in  $G$  and apply Equation 6.6 to see that Theorem 3.1(iv) is equivalent to  $[\kappa^C, \kappa^L] = 0$  by Definition 5.6. Note that this is equivalent to [12, (6)] when  $k = \mathbb{R}$ .  $\square$

We are now ready to express the PBW property purely in cohomological terms. Recall that  $H^\bullet$  is the fixed space of representatives of elements in  $\text{HH}^\bullet(S, S\#G)$  defined in (5.5).

**Theorem 7.2.** *A quotient algebra  $\mathcal{H}_{\bar{\kappa}}$  is a Drinfeld orbifold algebra if and only if  $\mathcal{H}_{\bar{\kappa}}$  is isomorphic to  $\mathcal{H}_{\kappa}$  as a filtered algebra for some parameter  $\kappa$  satisfying*

- $\kappa$  is  $G$ -invariant,
- The linear part of  $\kappa$  is a cocycle in  $H^\bullet$ ,
- The Gerstenhaber square bracket of the linear part of  $\kappa$  satisfies  $[\kappa^L, \kappa^L] = 2d^*(\kappa^C)$  as cochains,

- (iv) *The bracket of the linear with the constant part of  $\kappa$  is zero:  $[\kappa^C, \kappa^L] = 0$  as a cochain.*

*Proof.* Write  $\tilde{\kappa} = \tilde{\kappa}^L + \tilde{\kappa}^C$ . If  $\mathcal{H}_{\tilde{\kappa}}$  is a Drinfeld orbifold algebra, then it satisfies Condition (ii) of Theorem 3.1. Lemma 7.1 then implies that  $\tilde{\kappa}^L$  is a cocycle in  $\mathrm{HH}^\bullet(S, S \# G)$  expressed with respect to the Koszul resolution, and it thus lies in the set of cohomology representatives  $(H^\bullet)^G$  up to a coboundary:

$$\tilde{\kappa}^L = \kappa^L + d^* \rho$$

for some 2-cocycle  $\kappa^L$  in  $(H^2)^G$  and some 1-cochain  $\rho$ . Set

$$\kappa^C = \tilde{\kappa}^C + \rho \circ \tilde{\kappa}^L - \rho \otimes \rho$$

where  $(\rho \otimes \rho)(v \wedge w) := \rho(v)\rho(w) - \rho(w)\rho(v)$  for all  $v, w \in V$ . Let  $\kappa = \kappa^C + \kappa^L$ .

We may assume without loss of generality that  $\rho$  is  $G$ -invariant. (Note that  $d^* \rho = \tilde{\kappa}^L - \kappa^L$  is  $G$ -invariant. Since  $d^*$  commutes with the group action and the order of  $G$  is invertible in  $k$ , we may replace  $\rho$  by  $\frac{1}{|G|} \sum_{g \in G} {}^g \rho$  to obtain a cochain having the same image under  $d^*$ .) Also note that without loss of generality  $\rho$  takes values in  $kG$  since  $d^* \rho$  has polynomial degree 1.

Define a map  $f : T(V) \# G \rightarrow \mathcal{H}_\kappa$  by

$$f(v) = v + \rho(v), \quad f(g) = g$$

for all  $v \in V, g \in G$ ; since  $\rho$  is  $G$ -invariant, these values extend uniquely to give an algebra homomorphism. Note that  $f$  is surjective by an inductive argument on the degrees of elements.

We show first that the kernel of  $f$  contains the ideal

$$(vw - wv - \tilde{\kappa}^L(v, w) - \tilde{\kappa}^C(v, w) \mid v, w \in V),$$

which implies that  $f$  induces an algebra homomorphism from  $\mathcal{H}_{\tilde{\kappa}}$  onto  $\mathcal{H}_\kappa$ . By the definition of  $f$ ,

$$\begin{aligned} & f(vw - wv - \tilde{\kappa}^L(v, w) - \tilde{\kappa}^C(v, w)) \\ &= (v + \rho(v))(w + \rho(w)) - (w + \rho(w))(v + \rho(v)) \\ &\quad - \tilde{\kappa}^L(v, w) - \rho(\tilde{\kappa}^L(v, w)) - \tilde{\kappa}^C(v, w) \\ &= vw - wv + v\rho(w) + \rho(v)w - w\rho(v) - \rho(w)v + \rho(v)\rho(w) - \rho(w)\rho(v) \\ &\quad - \tilde{\kappa}^L(v, w) - \rho \circ \tilde{\kappa}^L(v, w) - \tilde{\kappa}^C(v, w) \\ &= vw - wv + d^* \rho(v \wedge w) + (\rho \otimes \rho)(v \wedge w) - \tilde{\kappa}^L(v, w) - \rho \circ \tilde{\kappa}^L(v, w) - \tilde{\kappa}^C(v, w) \\ &= vw - wv - \kappa^L(v, w) - \kappa^C(v, w) = 0 \end{aligned}$$

in  $\mathcal{H}_\kappa$ . Thus the ideal generated by all  $vw - wv - \tilde{\kappa}^L(v, w) - \tilde{\kappa}^C(v, w)$  is in the kernel of  $f$ .



Next we define an inverse to  $f$  by replacing  $\rho$  with  $-\rho$ : Define an algebra homomorphism  $f' : T(V) \# G \rightarrow \mathcal{H}_{\tilde{\kappa}}$  by  $f'(v) = v - \rho(v)$ ,  $f(g) = g$  for all  $v \in V$ ,  $g \in G$ . We have  $\kappa^L = \tilde{\kappa}^L - d^* \rho$  and

$$\begin{aligned} \tilde{\kappa}^C &= \kappa^C - \rho \circ \tilde{\kappa}^L + \rho \otimes \rho \\ &= \kappa^C - \rho \circ \kappa^L - \rho \circ (d^* \rho) + \rho \otimes \rho. \end{aligned}$$

Extending  $\rho$  in the usual way from a function on  $V$  to a function on  $V \otimes kG$  by setting  $\rho(vg) := \rho(v)g$  for all  $v \in V$ ,  $g \in G$ , we calculate

$$\begin{aligned} \rho \circ (d^* \rho)(v \wedge w) &= \rho(v\rho(w) + \rho(v)w - w\rho(v) - \rho(w)v) \\ &= \rho(v)\rho(w) + \rho(v)\rho(w) - \rho(w)\rho(v) - \rho(w)\rho(v) \\ &= 2(\rho \otimes \rho)(v \wedge w), \end{aligned}$$

since  $\rho$  has image in  $kG$ . Thus we may rewrite

$$\begin{aligned} \tilde{\kappa}^C &= \kappa^C - \rho \circ \kappa^L - 2\rho \otimes \rho + \rho \otimes \rho \\ &= \kappa^C - \rho \circ \kappa^L - \rho \otimes \rho \\ &= \kappa^C + (-\rho) \circ \kappa^L - (-\rho) \otimes (-\rho). \end{aligned}$$

An argument similar to that above for  $f$  (replacing  $\rho$  by  $-\rho$ ) shows that the function  $f'$  induces an algebra homomorphism from  $\mathcal{H}_{\kappa}$  onto  $\mathcal{H}_{\tilde{\kappa}}$ . By its definition,  $f'$  is inverse to  $f$ . Therefore  $\mathcal{H}_{\kappa}$  and  $\mathcal{H}_{\tilde{\kappa}}$  are isomorphic as filtered algebras. (Note that this isomorphism did not require that  $\mathcal{H}_{\tilde{\kappa}}$  satisfy the PBW condition, only that  $\tilde{\kappa}$  be a cocycle.) As  $\text{gr } \mathcal{H}_{\kappa} \cong \text{gr } \mathcal{H}_{\tilde{\kappa}} \cong S \# G$ , the quotient algebra  $\mathcal{H}_{\kappa}$  is also a Drinfeld orbifold algebra. Theorem 3.1 and Lemma 7.1 then imply the four conditions of the theorem.

Conversely, assume  $\mathcal{H}_{\tilde{\kappa}}$  is isomorphic, as a filtered algebra, to some  $\mathcal{H}_{\kappa}$  satisfying the four conditions of the theorem. Then Theorem 3.1 and Lemma 7.1 imply that  $\mathcal{H}_{\kappa}$  is a Drinfeld orbifold algebra. As the isomorphism preserves the filtration,

$$\text{gr } \mathcal{H}_{\tilde{\kappa}} \cong \text{gr } \mathcal{H}_{\kappa} \cong S(V) \# G$$

as algebras, and hence  $\mathcal{H}_{\tilde{\kappa}}$  is a Drinfeld orbifold algebra as well. Note that Theorem 6.3 shows that the bracket formula in the statement of the theorem indeed coincides with the Gerstenhaber bracket on cohomology.  $\square$

**Remark 7.3.** We compare the above results to Gerstenhaber's original theory of deformations, since every Drinfeld orbifold algebra defines a deformation of  $S \# G$  (see Section 2). The theory of Hochschild cohomology provides necessary conditions for "parameter maps" to define a deformation. Given a  $k$ -algebra  $R$  and arbitrary  $k$ -linear maps  $\mu_1, \mu_2 : R \otimes R \rightarrow R$ , we say  $\mu_1$  and  $\mu_2$  *extend* to first and second order approximations, respectively, of a deformation  $R[t]$  of  $R$  over  $k[t]$  if there are  $k$ -linear maps  $\mu_i : R \otimes R \rightarrow R$  ( $i \geq 3$ ) for which the multiplication in  $R[t]$  satisfies

$$r * s = rs + \mu_1(r \otimes s)t + \mu_2(r \otimes s)t^2 + \mu_3(r \otimes s)t^3 + \dots$$

for all  $r, s \in R$ , where  $rs$  is the product in  $R$ . Associativity forces  $\mu_1$  to define a cocycle in  $\mathrm{HH}^2(R)$ ; in addition, its Gerstenhaber square bracket must be twice the differential applied to  $\mu_2$ :

$$[\mu_1, \mu_1] = 2\delta_3^* \mu_2 .$$

Indeed, by using (2.3) and (2.4), we find that the equation  $[\kappa^L, \kappa^L] = 2d^* \kappa^C$  is a consequence of the equation  $[\mu_1, \mu_1] = 2\delta_3^* \mu_2$ : The left side of Theorem 3.1(iii) is both equal to  $-\kappa^L, \kappa^L$  applied to  $v_1 \wedge v_2 \wedge v_3$  and to  $-\mu_1, \mu_1$  applied to  $v_1 \wedge v_2 \wedge v_3$  by our previous analysis, identifying  $\alpha$  in the Braverman-Gaitsgory approach with the restriction of  $\mu_1$  to the space of relations  $R$ . The right side of Theorem 3.1(iii) is both equal to  $-2d^* \kappa^C$  applied to  $v_1 \wedge v_2 \wedge v_3$  and to  $-2\delta_3^* \mu_2$  applied to  $v_1 \wedge v_2 \wedge v_3$  since  $\mu_2 \circ \phi_2 = \kappa^C$  and  $\phi$  is a chain map (see (6.4)).

The square bracket  $[\mu_1, \mu_1]$  is called the *primary obstruction* to integrating a map  $\mu_1$  to a deformation: If a deformation exists with first-order approximation  $\mu_1$ , then  $[\mu_1, \mu_1]$  is a coboundary, i.e., defines the zero cohomology class of the Hochschild cohomology  $\mathrm{HH}^3(R)$ .

The parameter maps  $\kappa^L$  and  $\kappa^C$  (arising from the Koszul resolution) play the role of the first and second order approximation maps  $\mu_1$  and  $\mu_2$  (arising from the bar complex). We see in the proof of Theorem 2.1 that each  $\kappa_g^L g$  is in fact a *cocycle* when  $\mathcal{H}_\kappa$  is a Drinfeld orbifold algebra, and each  $\kappa_g^C g$  defines a second order approximation to the deformation. In fact, we expect  $\kappa^L$  to be invariant whenever  $\mathcal{H}_\kappa$  is a Drinfeld orbifold algebra since  $\mathrm{HH}^\bullet(S\#G) \cong \mathrm{HH}^\bullet(S, S\#G)^G$ . Note however that the theorem above goes beyond these elementary observations and Gerstenhaber's original formulation, which only give necessary conditions.

We now apply Theorem 7.2 in special cases to determine Drinfeld orbifold algebras from the set of necessary and sufficient conditions given in that theorem (in terms of Gerstenhaber brackets).

Recall that the Lie orbifold algebras are exactly the PBW algebras  $\mathcal{H}_\kappa$  in which the linear part of the parameter  $\kappa$  is supported on the identity group element  $1_G$  alone. Interpreting Proposition 4.2 in homological language, we obtain necessary and sufficient conditions for  $\kappa$  to define a Lie orbifold algebra in terms of the Gerstenhaber bracket:

**Corollary 7.4.** *Assume  $\kappa^L$  is supported on  $1_G$ . Then  $\mathcal{H}_\kappa$  is a Lie orbifold algebra if and only if*

- (a)  $\kappa^L$  is a Lie bracket on  $V$ ,
- (b) both  $\kappa^L$  and  $\kappa^C$  are  $G$ -invariant cocycles (define elements of  $\mathrm{HH}^2(S\#G)$ ),
- (c)  $[\kappa^C, \kappa^L] = 0$  as a cochain.

*Proof.* First note that  $[\kappa^L, \kappa^L] = 0$  exactly when  $\kappa^L$  defines a Lie bracket on  $V$ . Suppose Conditions (a), (b), and (c) hold. Condition (b) implies parts (i) and

(ii) of Theorem 7.2. It also implies that  $d^*(\kappa^C) = 0$ . Condition (a) implies that  $[\kappa^L, \kappa^L] = 0$ , and part (iii) of Theorem 7.2 is satisfied as well. Condition (c) is part (iv) of Theorem 7.2. Hence, Theorem 7.2 implies that  $\mathcal{H}_\kappa$  is a Lie orbifold algebra.

Conversely, assume that  $\mathcal{H}_\kappa$  is a Lie orbifold algebra. By Proposition 4.2,  $\kappa^L$  defines a Lie bracket on  $V$  and hence  $[\kappa^L, \kappa^L] = 0$ . Theorem 7.2 then not only implies Condition (c), but also that  $\kappa^L$  and  $\kappa^C$  are both cocycles with  $\kappa$   $G$ -invariant. But  $\kappa$  is  $G$ -invariant if and only if both  $\kappa^L, \kappa^C$  are  $G$ -invariant. Hence, Condition (b) holds.  $\square$

Recall that  $(H^\bullet)^G \cong \mathrm{HH}^\bullet(S\#G)$  and that  $C^\bullet$  and  $H^\bullet$  are sets of cochains and cohomology representatives, respectively (see (5.4) and (5.5)). Given  $\kappa^C, \kappa^L$  in  $(H^2)^G$  of homogeneous polynomial degrees 0 and 1, respectively, the sum  $\kappa := \kappa^C + \kappa^L$  is a parameter function  $V \wedge V \rightarrow (k \oplus V) \otimes kG$  defining a quotient algebra  $\mathcal{H}_\kappa$ . The last result implies immediately that for  $\kappa^L$  supported on  $1_G$ , the algebra  $\mathcal{H}_\kappa$  is a Lie orbifold algebra when  $\kappa^L$  is a Lie bracket on  $V$  and the cochain  $[\kappa^C, \kappa^L]$  is zero on the Koszul resolution. The hypothesis that  $\kappa^L$  be a Lie bracket is not as restrictive as one might think. In fact, if  $\kappa^L$  is a noncommutative Poisson structure (i.e., with Gerstenhaber square bracket  $[\kappa^L, \kappa^L]$  zero in cohomology), then  $\kappa^L$  is automatically a Lie bracket, as we see in the next corollary.

**Corollary 7.5.** *Suppose a linear cochain  $\kappa^L$  in  $C^2$  is supported on the kernel of the representation  $G \rightarrow \mathrm{GL}(V)$  and that  $[\kappa^L, \kappa^L]$  is a coboundary. Then  $[\kappa^L, \kappa^L] = 0$  as a cochain.*

*Proof.* Suppose  $[\kappa^L, \kappa^L] = d^*\alpha$  for some  $\alpha$ . Then (by definition of the map  $d^*$ ),

$$d_3^*\alpha(v_1 \wedge v_2 \wedge v_3) = - \sum_{g \in G} \sum_{\sigma \in \mathrm{Alt}_3} \alpha_g(v_{\sigma(2)}, v_{\sigma(3)}) ({}^g v_{\sigma(1)} - v_{\sigma(1)})g$$

for all  $v_1, v_2, v_3$  in  $V$ , and thus  $d^*\alpha$  is supported off the kernel  $K$  of the representation  $G \rightarrow \mathrm{GL}(V)$ . But by Definition 5.6,  $[\kappa^L, \kappa^L]$  is supported on  $K$ , since  $\kappa^L$  itself is supported on  $K$ . Hence  $[\kappa^L, \kappa^L]$  must be the zero cochain.  $\square$

The last corollary implies that every linear noncommutative Poisson structure supported on group elements acting trivially lifts (or integrates) to a deformation of  $S\#G$ :

**Corollary 7.6.** *Suppose a linear cocycle  $\kappa^L$  in  $(H^2)^G$  has trivial Gerstenhaber square bracket in cohomology. If  $\kappa^L$  is supported on the kernel of the representation  $G \rightarrow \mathrm{GL}(V)$ , then the quotient algebra  $\mathcal{H}_\kappa$  with  $\kappa = \kappa^L$  is a Drinfeld orbifold algebra. Moreover, if  $G$  acts faithfully on  $V$ , then  $\mathcal{H}_\kappa \cong \mathcal{U}(\mathfrak{g})\#G$ , a Lie orbifold algebra.*

*Proof.* Since  $\kappa^L$  lies in  $(H^2)^G$ , we may set  $\kappa^C \equiv 0$  and  $\kappa := \kappa^L$  to satisfy the conditions of Theorem 7.2 (using Corollary 7.5 to deduce that  $\kappa^L$  is a Lie bracket).

If  $G$  acts faithfully, the resulting Drinfeld orbifold algebra is just the skew group algebra  $\mathcal{U}(\mathfrak{g})\#G$ , where the Lie algebra  $\mathfrak{g}$  is the vector space  $V$  with Lie bracket  $\kappa^L$ .  $\square$

**Remark 7.7.** The analysis of the Gerstenhaber bracket in [21] includes information on the case of cocycles supported *off* the kernel  $K$  of the representation  $G \rightarrow \mathrm{GL}(V)$ . Indeed, we see in [21] that if  $\kappa^L$  in  $(H^2)^G$  is supported off  $K$ , then  $[\kappa^L, \kappa^L]$  is always a coboundary. This guarantees existence of a constant cochain  $\kappa^C$  with  $[\kappa^L, \kappa^L] = 2d^*\kappa^C$ . Thus to satisfy the conditions of Theorem 7.2, one need only check that  $[\kappa^C, \kappa^L] = 0$  as a cochain (on the Koszul resolution).

On the other hand, if  $\kappa^L$  in  $H^\bullet$  is supported *on* the kernel  $K$ , and  $[\kappa^L, \kappa^L]$  is a coboundary, then by Corollary 7.5,  $[\kappa^L, \kappa^L] = 0$  as a cochain. Thus to satisfy the conditions of Theorem 7.2, one need only solve the equation  $[\kappa^C, \kappa^L] = 0$  as a cochain for  $\kappa^C$  a *cocycle*.

## 8. APPLICATIONS TO ABELIAN GROUPS

The last section expressed the PBW condition in terms of simple conditions on Hochschild cocycles. We see in this section how this alternative formulation gives a quick and clear proof that every linear noncommutative Poisson structure (i.e., Hochschild 2-cocycle with trivial Gerstenhaber square bracket) lifts to a deformation when  $G$  is abelian. Halbout, Oudom, and Tang [12, Theorem 3.7] gave an analogous result over the real numbers for arbitrary groups (acting faithfully), but their proof does not directly extend to other fields such as the complex numbers. (For example, complex reflections in a finite group acting linearly on  $\mathbb{C}^n$  may contribute to Hochschild cohomology  $\mathrm{HH}^2(S\#G)$  defined over the real numbers, but not to the same cohomology defined over the complex numbers.)

In the case of nonabelian groups, the square bracket of the linear part of the parameter  $\kappa$  may be zero in cohomology but nonzero as a cochain. The following proposition explains that this complication disappears for abelian groups:

**Proposition 8.1.** *Let  $G$  be an abelian group. Let  $\alpha, \beta$  in  $(H^2)^G$  be linear with Gerstenhaber bracket  $[\alpha, \beta]$  a coboundary (defining the zero cohomology class). Then  $[\alpha, \beta] = 0$  as a cochain.*

*Proof.* Let  $v_1, \dots, v_n$  be a basis of  $V$  on which  $G$  acts diagonally. If  $[\alpha, \beta]$  is nonzero at the chain level, then some summand of Definition 5.6 is nonzero for some triple  $v_1, v_2, v_3$ . Suppose without loss of generality that

$$w = \beta_h(v_3 \wedge \alpha_g(v_1 \wedge v_2))$$

is nonzero for some  $g, h$  in  $G$ .

Note that if  $g$  acts nontrivially on  $V$ , then  $v_1$  and  $v_2$  must span  $(V^g)^\perp$  and  $\alpha_g(v_1 \wedge v_2)$  lies in  $V^g$  as  $\alpha_g(v_1 \wedge v_2)$  is nonzero and  $\alpha_g \in H_g^2$ . Similarly, if  $h$  acts

nontrivially on  $V$ , then  $v_3$  and  $\alpha_g(v_1 \wedge v_2)$  must span  $(V^h)^\perp$  and  $w$  lies in  $V^h$ . (See the comments after (5.5) or [18, Lemma 3.6].)

Suppose first that *both*  $g$  and  $h$  act nontrivially on  $V$ . Then  $v_3$  and  $\alpha_g(v_1 \wedge v_2)$  are independent vectors in  $V^g \cap (V^h)^\perp$ , a subspace of the 2-dimensional space  $(V^h)^\perp$ . Thus  $(V^h)^\perp \subset V^g$  and  $v_1, v_2$  in  $(V^g)^\perp$  are fixed by  $h$ . As  $G$  is abelian and  $\alpha$  is  $G$ -invariant,  $\alpha_g = {}^h\alpha_g$  and

$$\alpha_g(v_1 \wedge v_2) = ({}^{h^{-1}}\alpha_g)(v_1 \wedge v_2) = {}^{h^{-1}}\left(\alpha_g({}^h v_1 \wedge {}^h v_2)\right) = {}^{h^{-1}}\left(\alpha_g(v_1 \wedge v_2)\right).$$

But then  $\alpha_g(v_1 \wedge v_2)$  is fixed by  $h$ , contradicting the fact that it lies in  $(V^h)^\perp$ .

We use the fact that the cochain map  $[\alpha, \beta]$  represents the zero cohomology class to analyze the case when either  $g$  or  $h$  acts trivially on  $V$ . Calculations show that the image of the differential  $d^*$  is supported on elements of  $G$  that do not fix  $V$  pointwise (see, for example, Section 7). Hence  $V^{hg} \neq V$  and either  $g$  or  $h$  acts nontrivially on  $V$ . Also note that the coefficient of  $gh$  in any image of the differential lies in  $(V^{gh})^\perp$ .

If  $h$  acts nontrivially on  $V$  but  $g$  fixes  $V$  pointwise, then  $w$  lies in  $(V^{gh})^\perp = (V^h)^\perp$ , contradicting the fact that  $w$  lies in  $V^h$  (as  $h$  acts nontrivially). If instead  $g$  acts nontrivially on  $V$  but  $h$  fixes  $V$  pointwise, we contradict the  $G$ -invariance of  $\beta$ : In this case,

$$\begin{aligned} w &= \beta_h(v_3 \wedge \alpha_g(v_1 \wedge v_2)) = ({}^{g^{-1}}\beta_h)(v_3 \wedge \alpha_g(v_1 \wedge v_2)) \\ &= {}^{g^{-1}}\left(\beta_h({}^g v_3 \wedge {}^g(\alpha_g(v_1 \wedge v_2)))\right) = {}^{g^{-1}}\left(\beta_h(v_3 \wedge (\alpha_g(v_1 \wedge v_2)))\right) = {}^{g^{-1}}w \end{aligned}$$

(since both  $v_3$  and  $\alpha_g(v_1 \wedge v_2)$  lie in  $V^g$ ), so  $w$  lies in  $V^g = V^{gh}$  instead of  $(V^{gh})^\perp$ .  $\square$

As a consequence of Lemma 7.1 and Proposition 8.1, we obtain the following:

**Corollary 8.2.** *Let  $G$  be an abelian group. Suppose  $\kappa^L$  in  $(H^2)^G$  is a linear cocycle with  $[\kappa^L, \kappa^L]$  a coboundary. Then  $[\kappa^L, \kappa^L] = 0$  as a cochain. Thus we obtain a Drinfeld orbifold algebra  $\mathcal{H}_\kappa$  after setting  $\kappa^C \equiv 0$  and  $\kappa := \kappa^L$ .*

Other Drinfeld orbifold algebras with the same parameter  $\kappa^L$  arise from solving the equation  $[\kappa^C, \kappa^L] = 0$  for  $\kappa^C$  a cocycle of polynomial degree 0 in  $(H^2)^G$ . Compare with [12, Theorem 3.4], which is stated in the case that the action is faithful.

We end this section by pointing out a much stronger statement than that implied by [21, Theorem 9.2] for abelian groups: There we proved that for all groups  $G$ , the bracket of any two Hochschild 2-cocycles supported off the kernel of the representation is a coboundary (i.e., zero in cohomology). The proposition below (cf. [12, Lemma 3.3]) explains that when  $G$  is abelian, such brackets are not only coboundaries, they are zero as cochains.

**Proposition 8.3.** *Let  $G$  be an abelian group. Let  $\alpha, \beta$  in  $(H^2)^G$  be two linear Hochschild 2-cocycles on  $S\#G$  supported off of the kernel of the representation  $G \rightarrow GL(V)$ . Then  $[\alpha, \beta] = 0$  as a cochain.*

*Proof.* This statement follows immediately from [21, Theorem 9.2] and Proposition 8.1. However, we give a short, direct proof here: Let  $v_1, \dots, v_n$  be a basis of  $V$  on which  $G$  acts diagonally. If  $[\alpha, \beta]$  is nonzero, then some summand

$$\beta_h(v_3 \wedge \alpha_g(v_1 \wedge v_2))$$

of Definition 5.6 is nonzero for some triple  $v_1, v_2, v_3$  in  $V$  and some  $g$  and  $h$  in  $G$ . Since  $g$  and  $h$  both act nontrivially on  $V$ , the vector  $\alpha_g(v_1 \wedge v_2)$  must be invariant under  $h$  (as we saw in the third paragraph of the proof of Proposition 8.1). But this contradicts the fact that  $v_3$  and  $\alpha_g(v_1 \wedge v_2)$  must span  $(V^h)^\perp$ .  $\square$

One may apply Proposition 8.3 to find many examples of Drinfeld orbifold algebras of the type given in Example 3.3.

## REFERENCES

- [1] A. Beilinson, V. Ginzburg, and W. Soergel, “Koszul duality patterns in representation theory,” J. Amer. Math. Soc. 9 (1996), no. 2, 473–527.
- [2] G. Bergman, “The diamond lemma for ring theory,” Adv. Math. 29 (1978) no. 2, 178–218.
- [3] A. Braverman and D. Gaitsgory, “Poincaré-Birkhoff-Witt Theorem for quadratic algebras of Koszul type,” J. Algebra 181 (1996), 315–328.
- [4] J. L. Bueso and J. Gómez-Torrecillas and A. Verschoren, *Algorithmic methods in non-commutative algebra*. Applications to quantum groups. Mathematical Modelling: Theory and Applications, 17, Kluwer Academic Publishers, Dordrecht, 2003.
- [5] A. Căldăraru, A. Giaquinto, and S. Witherspoon, “Algebraic deformations arising from orbifolds with discrete torsion,” J. Pure Appl. Algebra 178 (2003), 87–100.
- [6] V. G. Drinfeld, “Degenerate affine Hecke algebras and Yangians,” Funct. Anal. Appl. 20 (1986), 58–60.
- [7] Y. Du, X. Chen, and Y. Ye, “On graded bialgebra deformations,” Algebra Colloq. 14 (2007), no. 2, 301–312.
- [8] P. Etingof and V. Ginzburg, “Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism,” Invent. Math. 147 (2002), no. 2, 243–348.
- [9] M. Farinati, “Hochschild duality, localization, and smash products,” J. Algebra 284 (2005), no. 1, 415–434.
- [10] V. Ginzburg and D. Kaledin, “Poisson deformations of symplectic quotient singularities,” Adv. Math. 186 (2004), no. 1, 1–57.
- [11] S. Griffeth, “Towards a combinatorial representation theory for the rational Cherednik algebra of type  $G(r, p, n)$ ,” Proc. Edinburgh Math. Soc. 53 (2010), 419–445.
- [12] G. Halbout, J.-M. Oudom, and X. Tang, “Deformations of orbifolds with noncommutative linear Poisson structures,” Int. Math. Res. Not. (2011), no. 1, 1–39.
- [13] A. Khare, “Drinfeld-Hecke algebras over cocommutative algebras,” arxiv: 0705.2067.
- [14] V. Levandovskyy and A. Shepler, “Quantum Hecke Algebras”, arXiv:1111.4975, submitted.
- [15] G. Lusztig, “Cuspidal local systems and graded Hecke algebras. I,” Inst. Haute Etudes Sci. Publ. Math. 67 (1988), 145–202.

- [16] G. Lusztig, “Affine Hecke algebras and their graded version,” J. Amer. Math. Soc. 2 (1989), no. 3, 599–635.
- [17] A. Ram and A.V. Shepler, “Classification of graded Hecke algebras for complex reflection groups,” Comment. Math. Helv. 78 (2003), 308–334.
- [18] A.V. Shepler and S. Witherspoon, “Hochschild cohomology and graded Hecke algebras,” Trans. Amer. Math. Soc. 360 (2008), no. 8, 3975–4005.
- [19] A.V. Shepler and S. Witherspoon, “Finite groups acting linearly: Hochschild cohomology and the cup product,” Adv. Math. 226 (2011), 2884–2910.
- [20] A.V. Shepler and S. Witherspoon, “Quantum differentiation and chain maps of bimodule complexes,” Algebra and Number Theory 5-3 (2011), 339–360.
- [21] A.V. Shepler and S. Witherspoon, “Group actions on algebras and the graded Lie structure of Hochschild cohomology,” arXiv:0911.0938, to appear in Journal of Algebra.
- [22] D. Ştefan, “Hochschild cohomology on Hopf Galois extensions,” J. Pure Appl. Algebra 103 (1995), 221–233.
- [23] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Adv. Math. 38, Cambridge Univ. Press, Cambridge, 1994.
- [24] S. Witherspoon, “Twisted graded Hecke algebras,” J. Algebra 317 (2007), 30–42.

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